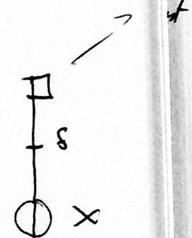
Ralf 11/3

Theorem

MM $\Rightarrow$ $\Pi_2^{H_{\omega_2}}$-maximality.

Def. Let ϕ be a statement. Then ϕ is 1-Ω-consistent iff $\forall x_0 \in \mathbb{R}$ there is a transitive model $\mathcal{M} = (M; \in)$ s.t.

- $\mathcal{M} \models ZFC + \phi$
- $x_0 \in M$ and $\mathcal{M}$ is closed under $X \mapsto M_1^\#(X)$

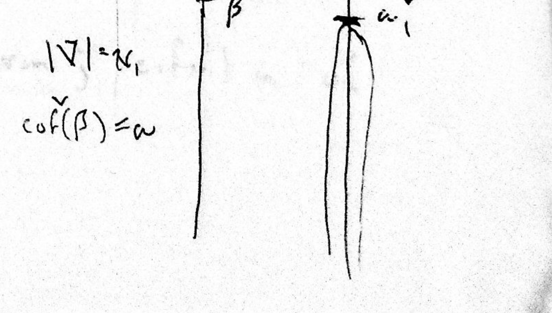


Def. Let ϕ be a statement, and let $A \in V$. $\phi(A)$ is 1-honestly-consistent iff inside $V^{Col(\omega, 2^{\aleph_1})}$ there is a transitive model $\mathcal{M}$ s.t.

- $\mathcal{M} \models ZFC + \phi(A)$
- $H_{\omega_2}^V \in M$
- $NS_{\omega_1}^{\mathcal{M}} \cap V = NS_{\omega_1}^V$,
- M is closed under $X \mapsto M_1^\#(X)$.

- $\Pi_2^{H_{\omega_2}}$-maximality: if ϕ is $\Pi_2^{H_{\omega_2}}$, and if ϕ is 1-Ω-consistent, then ϕ is true.

Lemma Let ϕ be Σ_1, and let $A \in H_{\omega_2}$. Suppose $\phi(A)$ is 1-honestly-consistent. Then so is $\phi(A) \wedge cof(\omega_2^V) = \omega$.



Proof

$\mathcal{M} \models ZFC \wedge \Phi(A)$

Remark: lemma requires closing a bit under $X \mapsto M_2^\#(X)$.

$H_{\omega_2}^{\check V} \in \mathcal{M}$

$\mathcal{M} \models \mathrm{cof}(\omega_2^{\check V}) = \omega$ $\qquad (\beta = (\omega_2^{\#})^{\check V})$

$\check V \subseteq_1 \cancel{Col(\omega, (2^{\aleph_1})^{\check V})}\ V^{Col(\omega, (2^{\aleph_1})^{\check V})}$

$\Rightarrow V \models$ there is a model like $\mathcal{M}$, replacing $\check V$ with V by absoluteness. $\square$

Lemma: Suppose NS_{ω_1} is saturated (in V).

Suppose Φ is Σ_1, $A \in H_{\omega_2}$, $\Phi(A) \wedge \mathrm{cof}(\omega_2^V) = \omega$ is 1-honestly consistent as witnessed by the model $\mathcal{M} \in V^{Col(\omega, 2^{\aleph_1})}$.

Then in $\mathcal{M}$ there is a generic iteration $(M_i, \pi_{ij} : i \leq j \leq \omega_1)$ with

M_0 is ctbl and $M_{\omega_1} = (H_{\omega_2}^V, \in, NS_{\omega_1}^V)$

——— " ———

$M_i = (M_i, \in, I_i)$

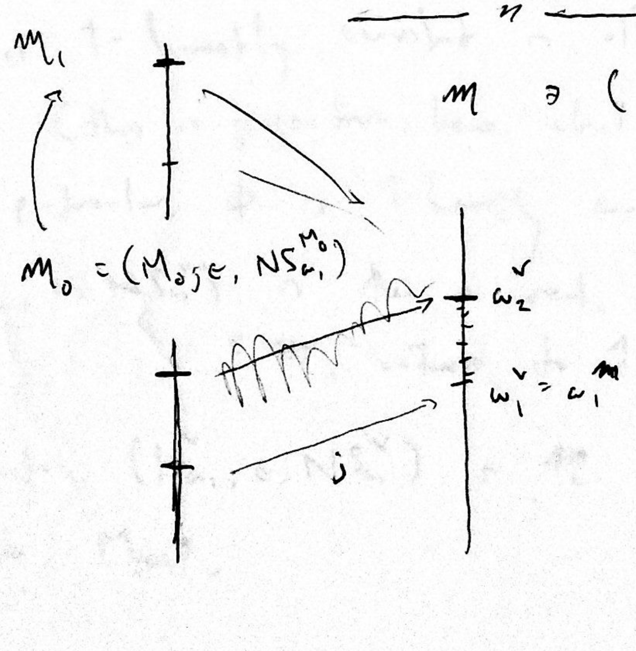
$\uparrow$

$NS_{\omega_1}^{M_i}$

Iterate by generic ultrapowers mapping I_i to I_{i+1}

——— " ———

$\mathcal{M} \ni (H_{\omega_2}^V, \in, NS_{\omega_1}^V)$



ran j cofinal in ω_2^V

g $\mathcal{U} = \{ S \in P(\omega_1^{M_0}) \cap M_0 : \omega_1^{M_0} \in j(S) \}$

<u>Key fact</u>: g generic for $(NS_{\omega_1}^{M_0})^+$ - generic over M_0

Pf. Let $A \in M_0$ be a maximal antichain in $(NS_{\omega_1}^{M_0})^+$. ~~[illegible]~~

A is sealed.

$(S_i : i < \omega_1)$, C club s.t. $\forall \alpha \in C$ $\alpha \in S_i$ for some $i < \alpha$.

$\alpha_0 = CRT(j) = \omega_1^{M_0}$

If C is in the range, then $\alpha_0 \in C$.

$\Rightarrow$ $\alpha_0 \in S_i$, some $i < \alpha_0$.

$\Rightarrow$ $S_i \in \operatorname{ran} j$.

$\alpha_0 \in j(S_i \cap \alpha_0) \in A$. $\square$

~~[illegible]~~

<u>Proof of theorem</u>

Fix $\phi \equiv \forall X \in H_{\omega_2} \exists Y \in H_{\omega_2} \Phi(X, Y)$, ϕ 1-Ω-consistent.

WTS ϕ is true

Fix $A \in H_{\omega_2}$. NTS $\exists Y\, \Phi(A, Y)$.

ϕ is 1-honestly consistent in all generic extensions

(this is projective, hence absolute)

In particular, ϕ is 1-honestly consistent in $V^{Col(\omega_1, \omega_2)}$.

$(H_{\omega_2}^V, \in, NS_{\omega_1}^V)$ is then a real.

$\in \mathcal{M}$, witness to 1-Ω-consistency of ϕ.

Iterate $(H_{\omega_2}^V, \in, NS_{\omega_1}^V)$ in $\mathcal{M}$ in iteration of length $(\omega_1)^{\mathcal{M}}$

to $M_{(\omega_1)^{\mathcal{M}}}$.

$$(H_{\omega_2}^V, \in, NS_{\omega_1}^V) \xrightarrow[j]{\text{generic iteration}} (M_{\omega_1}^{\mathcal{M}}, \in, NS_{\omega_1}^{\mathcal{M}} \cap M_{\omega_1}^{\mathcal{M}})$$

$j(A)$

$\mathcal{M} \models \exists Y\, \Phi(A, Y)$

Can lift iteration to act on each initial segment of Y.

$$\begin{array}{ccc} V & \longrightarrow & V' \\ \cup & & \cup \\ (H_{\omega_2}^V, \in, NS_{\omega_1}^V) & \xrightarrow[j]{} & (M_{\omega_1}^{\mathcal{M}}, \in, NS_{\omega_1}^{\mathcal{M}} \cap M_{\omega_1}^{\mathcal{M}}) \in \mathcal{M} \end{array}$$

$V' \models \exists Y\, \Phi(j(A), Y)$ (by absoluteness)

$V \models \exists Y\, \Phi(A, Y)$.

$\square$

<u>Key Lemma</u>. NS_{ω_1} saturated + V closed under $M_1^{\#}$.

If $\exists Y\, \Phi(A, Y)$ is 1-honestly consistent,

then $\exists$ ssp $\mathbb{P}$ s.t. $V^{\mathbb{P}} \models \exists Y\, \Phi(A, Y)$.