Ralf 11/6

Theorem
There is a ZFC model of $MM^{*,++}_{\aleph_2}$.

- ($\aleph_2 = 2^{\aleph_0}$)
- will disregard the "++" WLOG

How do get the ZFC model?

Let D be a determinacy model,

$D \models AD + DC + \Theta$ regular limit of Solovay sequence + every set of reals is uB + (some maximality hypotheses)

$D^{\mathbb{P}_{max} * Col(\omega_3, \omega_3)}$.

We want to verify $MM^*_{\aleph_2}$ holds in this $\mathbb{P}_{max}$ extension. Work in $D^{\mathbb{P}_{max}}$

Fix $\mathcal{M} = (M; \vec{R})$, $|M| = \aleph_2$.

$\aleph_1$-many; can be coded into 1 relation: $R = \{(x, \vec{y}) : x \in WO \wedge \vec{y} \in WO_{\|x\|}\}$

WMA $M = \mathbb{R}$, hence $\mathcal{M} = (\mathbb{R}; R)$.

Fix ϕ Σ_1. Assume $\phi(\mathcal{M})$ is honestly consistent, ie

for all $F: HC \to HC$ which is uB in the codes,

there is a transitive F-closed model $A \models ZFC^- + \phi(\mathcal{M})$ in $V^{Col(\omega, \aleph_2)}$

s.t. $H^V_{\omega_2} \in A$ and $NS^V_{\omega_1} = NS^A_{\omega_1} \cap V$.

To show: $\exists$ $\bar{\mathcal{M}} \hookrightarrow \mathcal{M}$, $\omega_1 \subset \bar{\mathcal{M}}$, $|\bar{\mathcal{M}}| = \aleph_1$, s.t. $\phi(\bar{\mathcal{M}})$.

R is essentially a set of reals. So in D, we have

$A := \{(q, \psi, \vec{z}) : q \Vdash_{\mathbb{P}_{max}} (\mathbb{R}; R) \models \psi(\vec{z})\} \subset {}^{<\omega}\mathbb{R}$

Def. (p, h) is good iff

(a) $p \in \mathbb{P}_{max}$

(b) $h \subseteq \mathbb{P}_{max}$ is a filter, $h \ni p$.

$h > p$ (i.e. $q > p$ $\forall q \in h$)

(a) Let $\mathbb{R}^h = \bigcup_{q \in h} \mathbb{R} \cap q$. Then $WO \cap \mathbb{R}^h = \omega_1^p$ and

$$h \Vdash (\mathbb{R}; R) \models \varphi(\vec{z}) \quad \forall \varphi, \ \forall \vec{z} \in \mathbb{R}^h$$

$\boxed{\text{(c) } A \cap p \in p \text{ and if } i : p \to p' \text{ comes from a ctbl generic iteration, then } i(A \cap p) = A \cap p'}$

and for all $\vec{z} \in \mathbb{R}^h$ there is some $y \in \mathbb{R}^h$ s.t.

$$h \Vdash \text{“}(\mathbb{R}; R) \models \text{“}\exists \vec{y}\, \varphi(\vec{y}, \vec{z}) \to \varphi(y, \vec{z})\text{''.''}$$

This "Henkin property" can be written informally:

$$\forall \varphi \forall \vec{z} \in \mathbb{R}^h \exists y \in \mathbb{R}^h \exists \tau \in h \ ((\tau, \ulcorner \exists \vec{y}\, \varphi(\vec{y}, \vec{v})\urcorner, \vec{z}) \in A \to (\tau, \ulcorner \varphi(\vec{y}, \vec{v})\urcorner, y^\frown \vec{z}))$$

(e) ~~p~~ $p \models \phi((\mathbb{R}^h; R^h))$, where

$$R^h = \{(x, \vec{y}) : x, \vec{y} \in \mathbb{R}^h, x \in WO, \exists q \in h \ (q, \text{“}\vec{v} \in \dot{R}\text{''}, (x, \vec{y})) \in A\}$$

~~Lemma~~ Fix g $\mathbb{P}_{max}$-generic / D.

~~Su~~

<u>Lemma</u> Let (p, h) be good, $p \in g$, ~~Then ... D...~~

Let $i : p \to p^*$ be the iteration of length ω_1 given by g.

Then $(\mathbb{R}^{i(h)}; R^{i(h)}) \prec (\mathbb{R}, R)$, and

$\phi((\mathbb{R}^{i(h)}; R^{i(h)}))$.

<u>Note</u>: $i(h) \in g$!

NTS there is a good pair in ~~the~~ $D^{\mathbb{P}_{max}} \times Col(\omega, \aleph_2)$

($\exists$ of good pairs is $\Sigma^1_3(A)$)

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Have $A \models \Phi((\mathbb{R};R))$, A closed under F.

Let X be witness to ϕ.

$M_1^\#(\mathbb{R}, R, g, X)^{Col(\omega,<\delta)}$ gives a $\mathbb{P}_{max}$ condition.

$(\underbrace{M_1^\#(\mathbb{R}, R, g, X)^{Col(\omega,<\delta)}, g}_{p})$ is a good pair

NTS ©: $A \cap p \in p$, p as above, and

if $p \to p'$ is a generic iteration then $i(A \cap p) = A \cap p'$

__Def__ ~~A premouse N captures A if~~

$(N, \delta, \tau, \Sigma)$ __captures__ A iff

- N is a ctbl pm $\models$ δ is a Woodin cardinal
- τ ~~[illegible]~~ $\in N^{Col(\omega,\delta)}$
- Σ is an iteration strategy w/ branch condensation etc for N
- if $i: N \to N'$ by Σ, and $g \in V$ is $Col(\omega, i(\delta))$-gen/N', then $i(\tau)^g = A \cap N'[g]$.

- This gives a robust version of u-Bness of A.

Now use

$M_2^{\Sigma,\#}(\mathbb{R}, R, g, X)$

to capture A, and use it as a $\mathbb{P}_{max}$ condition in good pair

See Ralf's paper in the Woodin volume, "MM^{++}, Woodin's (*) cases, ... or both?"