- $S(\kappa)$ κ-Suslin sets of reals

  Closed under $\exists^{\mathbb{R}}$, $\wedge_\omega$, $\cup_\omega$

  Not nec under $\forall^{\mathbb{R}}$, $\neg$

  $S(\kappa)$ or $\check{S}(\kappa)$ has scales

- κ Suslin $\longleftrightarrow$ $S(\kappa) \neq \bigcup_{\alpha<\kappa} S(\alpha)$

Projective-like hierarchies

Reference: Jackson, "Structural consequences of AD"

Let κ be a limit of Suslin cardinals,

$\underset{\sim}{\Delta}_0 = \bigcup_{\alpha<\kappa} S(\alpha)$ closed under $\exists^{\mathbb{R}}$, $\forall^{\mathbb{R}}$, $\neg$

$\kappa = \delta(\underset{\sim}{\Delta}_0)$ = Wadge ordinal of $\underset{\sim}{\Delta}_0$

$\Sigma^0_1 \approx \bigcup_\omega \underset{\sim}{\Delta}_0$ $\quad$ $\bigwedge_\omega \underset{\sim}{\Delta}_0 \approx \Pi^0_1$

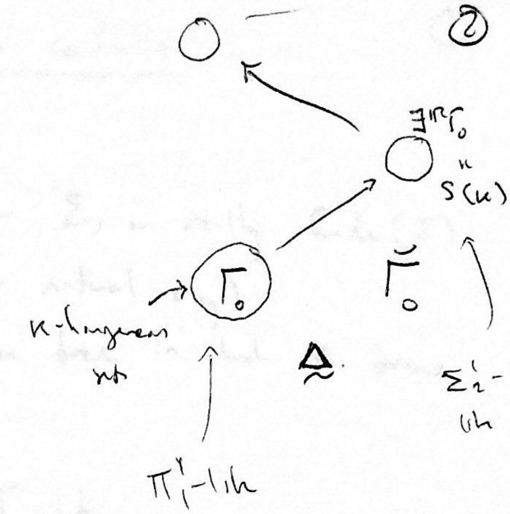
$\underset{\sim}{\Delta}_0$

Pattern of scales analogous to proj.

Type I cof $\kappa = \omega$

"Just like" projective hierarchy.

Type II cof $\kappa > \omega$, $\kappa > \operatorname{cof} \kappa$



$\Gamma_0 = \{p[T] : T \text{ on } \omega\times\kappa,\ T \text{ homogeneous}\}$

$\Downarrow$

$\forall^{\mathbb{R}}\Gamma_0 \subseteq \Gamma_0$

Type III κ regular, but Γ^0 is closed under ~~both $\exists^{\mathbb{R}}$~~, $\forall^{\mathbb{R}}$ but not $\exists^{\mathbb{R}}$.

Same diagram as Type II

Kleene pt class is example.

(Aside: Hyperprojective $\equiv$ recursive in $\exists^{\mathbb{R}}$)

Q (ZF) Are the κ-hom sets closed under $\forall^{\mathbb{R}}$?

Type IV Γ_0 closed under $\forall^{\mathbb{R}}$, $\exists^{\mathbb{R}}$.

(Inductive-like) + PWO

"Inaccessible from below"

Scales do not appear for a long while $\qquad \operatorname{Env}(\Gamma_0 \cup \breve{\Gamma}_0)$

$\operatorname{Env}(\Gamma_0 \cup \breve{\Gamma}_0)$: has reflection properties that say nothing dramatically new happens b/c still far from next scale.

$$\Gamma_0 = S(\kappa) \qquad \breve{\Gamma}_0$$

Gap in scales

Thm In $L(\mathbb{R})$, $\operatorname{Env}(\Sigma^2_1) = P(\mathbb{R})$.

(Boolean algebra over $\Gamma_0 \oplus \Gamma_1$ generates proj hierarchy & such $\forall^{\mathbb{R}}, \exists^{\mathbb{R}}$ generate new complexity

Mouse pairs + mouse limits ⇒ Suslin cardinals

Sps have mouse pair (P, Σ)

WTF an optimal Suslin representation for Σ, or really $\mathrm{Code}(\Sigma)$ (= set of reals coding $\Sigma \subseteq HC$ in natural way).

Do this by building a universal iteration tree in which all trees by Σ embed

Theorem $\exists!$ normal iteration tree T s.t.

① T is by Σ and ② T has last model $M_\infty(P, \Sigma)$.

↑

all countable "subtrees" are by Σ,

since T uncbl

Whenever $\mathcal{U}$ is a ctbl tree on P and there is a weak tree embedding $\Phi : \mathcal{U} \to T$, then $\mathcal{U}$ is by Σ.

(Rmk Any mouse pair (Q, Λ) is s.t. it has "very strong hull condensation", i.e. if T on (Q, Λ) and $\Phi : \mathcal{U} \to T$ is a weak tree embedding, then $\mathcal{U}$ is by ~~Σ~~ Λ.)

- Siskind, Steel. Full normalization for mouse pairs.

Let $\mathcal{U}(P, \Sigma)$ = the unique normal tree as above.

Cor $\mathrm{Code}(\Sigma)$ is $|M_\infty(P, \Sigma)|$-Suslin.

Pf.
Your tree S s.t. $p[S] = \mathrm{Code}(\Sigma)$ guesses T by building $\Phi : T \to \mathcal{U}(P, \Sigma)$. □

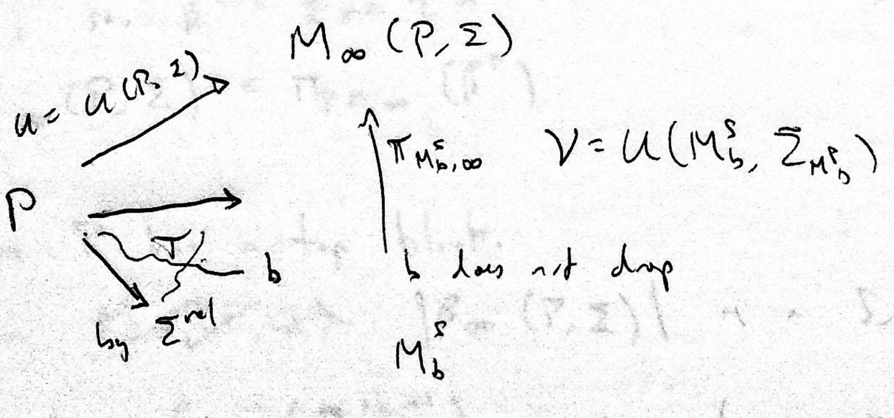
Rmk $\mathrm{Code}(\Sigma)$ is not α-Suslin $\forall \alpha < |M_\infty(P,\Sigma)|$.

Pf.
Kunen–Martin Theorem: Every α-Suslin wellfounded relation on $\mathbb{R}$ has rank $< \alpha^+$.
From Σ, the tree system give wellfounded relation of rank $|M_\infty(P,\Sigma)|$. □

Rmk
Not all cbH trees are used in generating $M_\infty(P,\Sigma)$!
E.g. $P \models \forall X$ X is cbH $\wedge$ $X^\#$ exists.
Iterating P induces drops.
The strategy for P can then be complicated.
So have to restrict to "relevant part" of Σ, Σ^{rel}.
For trees that can be extended by dropping on main branch
T s.t. $T \subseteq S$ where S does not drop

Rmk
No difference btw Σ, Σ^{rel} wrt pointclass generators.

Why does $\Sigma^{rel} = p[S]$?



$\mathcal{U} = X(T,\mathcal{V})$. Construction of X produces a weak tree embedding of $T \to X$.

Full Normalization

Another way to get a Suslin cardinal from M_∞'s:

E.g. $\aleph_\omega = o(M_\infty(M_1|\delta_1, \Sigma_{M_1|\delta_1}))$

$\omega_1 = |\beta_\infty(M_1|\delta_1, \Sigma_{M_1|\delta_1})|$, β_∞ = least strong of M_∞ to $o(M_\infty)$

$= \pi_{M_1|\delta_1,\infty}$(least strong to δ_1 of M_1)

<u>Significance of least strong</u>

Least strong is largest cut point,

hence iterating below vs above can be decoupled.

For any P, let $\tau^P = \sup\{\kappa^{+,P} : \kappa < \delta_{\text{lg}}(P) \wedge \exists E$ on P-sequence w/ $\text{crt}(E) = \kappa \wedge \text{lh}(E) > \kappa^{+,P}\}$

<u>Remark</u> Code(Σ) is $\tau_\infty(P,\Sigma)$-Suslin.

we only need to consult total extenders.

P has <u>least type</u> iff $\forall \alpha < \tau^P$ $o(\alpha)^P < \tau^P$

$o(\alpha)^P = \sup\{\text{lh}(E) : \text{crt}(E) = \alpha\}$

P has <u>top block</u> o.w.

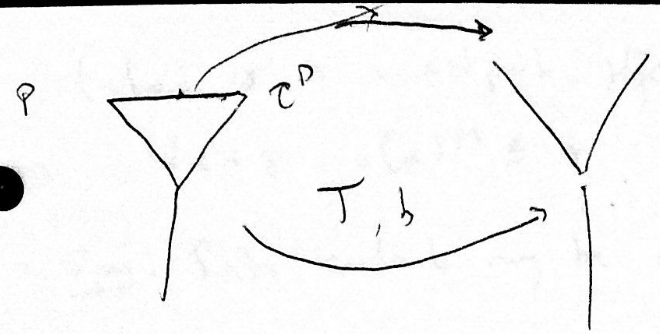
If so, $\beta^P = \min\{\alpha < \tau^P \mid o(\alpha)^P \not< \tau^P\}$.

$\beta_\infty(P,\Sigma) = \pi_{P,\Sigma,\infty}(\beta^P)$

Suppose P has a top block.

Then $\cancel{\tau_\infty(P,\Sigma)}$ $|\beta_\infty(P,\Sigma)|$ is a Suslin cardinal

Namely, Code($\Sigma^{\text{sh,rel}}$) is the witness, is $|\beta_\infty|$-Suslin and not α-Suslin $\forall \alpha < \beta_\infty$



$\delta(T) = \sup_{\alpha < \mathrm{lh}\, T} \mathrm{lh}\, E^T_\alpha$

T short iff $i_b(\tau) > \delta(T)$. iff T is normally properly continuable.

(Maximal trees only continuable via a stack.)

<u>Def</u> $\mathcal{U}^\circ(P,\Sigma) = \mathcal{U}(P,\Sigma) \upharpoonright \alpha+1$, ~~[illegible]~~
α least w/ $\mathrm{crit}(i^{\mathcal{U}(P,\Sigma)}_{\alpha,\infty}) > i^{\mathcal{U}(P,\Sigma)}_{0,\alpha}(\beta^P)$.

- T short, M_∞-rel, by Σ iff $\exists$ weak tree embedding $\Phi : T \to \mathcal{U}^\circ(P,\Sigma)$
- $|\mathcal{U}^\circ(P,\Sigma)| = \beta_\infty$

So $\mathrm{Code}(\Sigma^{\mathrm{sh,rel}}) = |\beta_\infty|$-Suslin

Kunen-Martin $\Rightarrow$ $\mathrm{Code}(\Sigma^{\mathrm{sh,rel}})$ not α-Suslin $\forall \alpha < |\beta_\infty|$.

<u>Thm</u> (Jackson, Sargsyan, S.)
AD^+. Let (P,Σ) be a mouse pair. Let $\kappa \leq o(M_\infty(P,\Sigma))$.
~~then~~ TFAE:
(1) κ is a Suslin cardinal.
(2) $\kappa = \gamma$ for γ a cutpoint of $M_\infty(P,\Sigma)$.

(where γ is a cutpoint ~~[illegible]~~ of M iff
$\forall \alpha < \gamma \quad o(\alpha)^M \leq \gamma$)

<u>Open</u>: Such cardinals may be the cutpoints, not their cardinalities.