1. Simon's Semester – Steel

24 X '23

$S(\kappa) := \{A \subseteq \mathbb{R} : A \text{ is } \kappa\text{-Suslin}\}$

κ is a Suslin card iff $S(\kappa) \neq \bigcup_{\alpha<\kappa} S(\alpha)$.

Projective-like hierarchies (Jackson handbook paper)

Let κ be a limit of Suslin cardinals,

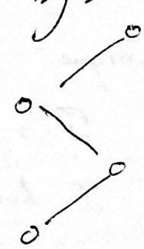
$\underset{\sim}{\Delta}_0 = \bigcup_{\alpha<\kappa} S\alpha$.

It is closed under $\exists^{\mathbb{R}}, \forall^{\mathbb{R}}, \neg$

$\kappa = \delta(\underset{\sim}{\Delta}_0) =$ Wadge ordinal of $\underset{\sim}{\Delta}$

We have two cases (of such hierarchies)

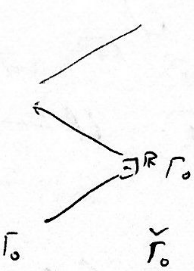
Type I $cf(\kappa_0) = \omega$ (hierarchy looks "just like" projective:



Type II $\omega < cf(\kappa) < \kappa$

$\Gamma_0 = \{p[T] : T \text{ on } \omega\times\kappa \text{ homogeneous}\}$

$\forall^{\mathbb{R}}\Gamma_0 \subseteq \Gamma_0$, but Γ_0 not closed under finite unions (?)



Type III κ regular, but Γ_0 is closed under ~~both $\exists^{\mathbb{R}}, \forall^{\mathbb{R}}$~~ $\forall^{\mathbb{R}}$, but not $\exists^{\mathbb{R}}$

· Pattern like I with bottom class with better closure properties.

2. Type IV P_0 is closed under both $\forall$, $\exists$.

(Inductive-like) + pwo-property.

next Suslin $S(\rho)$ is very long for up here in Wadge hierarchy

$End(\Gamma \cup P_0)$

$\Gamma_0 \check{P}_0$ → From Boole(P_0), you can build a whole projective hierarchy, before hitting next Suslin obs.

$S(x) = \Gamma_0 \quad \check{\Gamma}_0$

(This phenomenon is called <u>gap in scales</u>)

Mouse pairs + mod limits ⇒ Suslin cardinals.

Suppose we have a mouse pair (P, Σ). We want to find an optimal Suslin representation for it's strategy.

<u>Theorem</u>

There is a unique normal tree $\mathcal{T}$ on P s.t.

(1) $\mathcal{T}$ is by Σ

(2) $\mathcal{T}$ has last model $M_\infty(P, \Sigma)$. (club many elementary submodels are by Σ.

Here „$\mathcal{T}$ is by Σ" means: ~~every countable~~ whenever $\mathcal{U}$ is a ctbl tree on P and there is a ~~tree e~~ weak tree embedding $\Phi: \mathcal{U} \to \mathcal{T}$, then $\mathcal{U}$ is by Σ.

<u>Remark</u>

Any mouse pair (Q, Λ) has very strong hull condensation, i.e. if $\mathcal{T}$ on (Q, Λ) (on Q by Λ) and $\Phi: \mathcal{U} \to \mathcal{T}$ is a weak tree embedding, then $\mathcal{U}$ is by Λ.

(Siskind, S., Arxiv „Full normalization for mouse pairs"

<u>Definition</u> Let $\mathcal{U}(P, \Sigma)$ = the unique normal tree as above.

<u>Corollary</u> Code(Σ) is $|M_\infty(P, \Sigma)|$-Suslin.

3. Simon's Seminar (Steel) 24 X '23

Proof Your tree S s.t. $p[S] = \text{code}(\Sigma)$ justifies $\mathcal{T}$ by building $\Phi : \mathcal{T} \to \mathcal{U}(P, \Sigma)$.

$|M_\infty(P,\Sigma)|^{\nu}$ is Suslin, bc $\text{Code}(\Sigma)$ is not α-Suslin for $\alpha < |M_\infty(P,\Sigma)|$. Indeed, by Kunen–Martin, every α-Suslin wf relation on $\mathbb{R}$ has rank $< \alpha^+$.

Why $\Sigma^{rel} = p[S]$?

We don't literally look at $\text{Code}(\Sigma)$, only it's relevant part: the trees that can be extended so that they don't drop along the branch, all it Σ^{rel}

$$\begin{tikzcd} & M_\infty(P,\Sigma) \\ P \arrow[ur, "\mathcal{U}(P,\Sigma)"] \arrow[dr] & \\ & M_b^{\mathcal{T}} \arrow[uu, "\pi_{M_b^{\mathcal{T}}, \infty}"'] \end{tikzcd}$$

$M_b^{\mathcal{T}}$, ~~$\mathcal{T}$ to~~ b doesn't drop.

Let $\mathcal{V} = \mathcal{U}(M_b^{\mathcal{T}}, \Sigma_{M_b^{\mathcal{T}}})$, normalization

$\mathcal{U} = X(\mathcal{T}, \mathcal{V})$ – full normalization of stack $\langle \mathcal{T}, \mathcal{V} \rangle$.

construction of X, produces a weak tree embedding of $\mathcal{T} \to X$.

Another way to get a Suslin cardinal from M_∞:

e.g. $\lambda_\omega = \text{ord}(M_\infty(M_1 | \delta_1, \Sigma_{M_1|\delta_1}))$

$\omega_1 = |\beta_\infty(M_1|\delta_1, \Sigma_{M_1|\delta_1})|$, where

β_∞ = least strong of M_∞ to $\text{ord}(M_\infty)$.

$= \pi_{M_1|\delta_1, \infty}$ (least strong to δ_1 of M_1).

For any P, $\sigma^P = \sup \{ (\alpha^+)^P ; \alpha < \rho(P)$ & there exists E on P-sequence with $\text{crit}\, E = \alpha$ and $\text{lh}\, E > \alpha^+, P \}$

4. Definition $\mathcal{P}$ has limit type iff $\forall \alpha < \varepsilon^{\mathcal{P}}$

$o(\alpha)^{\mathcal{P}} < \delta^{\mathcal{P}}$

$\downarrow$

sup lh's of E with crit α.

$\mathcal{P}$ has top block otw.

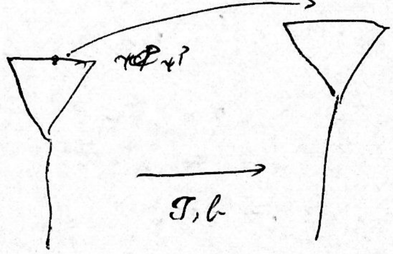
if so, $\beta^{\mathcal{P}}$ = least $\alpha < \delta^{\mathcal{P}}$ s.t. $o(\alpha)^{\mathcal{P}} = \delta^{\mathcal{P}}$.

$$\beta_\infty(\mathcal{P}, \Sigma) = \pi_{(\mathcal{P},\Sigma),\infty}(\beta^{\mathcal{P}})$$

Suppose $\mathcal{P}$ has a top block. Then $|\beta_\infty(\mathcal{P}, \Sigma)|$ is a Suslin cardinal.

Namely Code $(\Sigma^{short, rel})$ is $|\beta_\infty|$ - Suslin and not α-Suslin for $\alpha < \beta_\infty$.

$i_b(\lambda)$ $(\mathcal{P}) > \delta(\mathcal{T})$



| where $\mathcal{T}$ is short iff

| where $\delta(\mathcal{T})$ = sup lh $E_\alpha^{\mathcal{T}}$: $\alpha <$ lh $\mathcal{T}$,

so short tree is a one that can be continued with normal iteration.

Definition $\mathcal{U}^0(\mathcal{P}, \Sigma) = \mathcal{U}(\mathcal{P}, \Sigma) \restriction \alpha+1$, where α is least s.t. $cp\left(\lambda_{\alpha,\infty}^{\mathcal{U}(\mathcal{P},\Sigma)}\right) > i_{0\alpha}^{\mathcal{U}(\mathcal{P},\Sigma)}(\beta^{\mathcal{P}})$

$\mathcal{T}$ short, M_∞-relevant by Σ iff $\exists$ weak tree embedding

$$\underline{\Phi} : \mathcal{T} \longrightarrow \mathcal{U}^0(\mathcal{P}, \Sigma),$$

So Code $(\Sigma^{sh, rel})$ is $|\beta_\infty|$ - Suslin.

Kunen - Martin $\Rightarrow$ Code $(\Sigma^{sh, rel})$ is not α - Suslin for $\alpha < |\beta_\infty|$.

5. Simon's Semester (Steel) 24 X '23

<u>Theorem</u> (Jackson, Sargsyan, Steel) (AD^+)

Let (P,Σ) be a mouse pair. Let $\kappa \leq o(M_\infty(P,\Sigma))$.

~~be a δ~~ TFAE:

(1) κ is a Suslin cardinal.

(2) $\kappa = |\gamma|$, where γ is a cutpoint of $M_\infty(P,\Sigma)$

(where γ is a <u>cutpoint</u> of M iff $\forall \alpha < \gamma$

$$o(\alpha)^M \leq \gamma.$$

<u>Conjecture</u> Let (P,Σ) with a top block

$$|M_\infty(P,\Sigma)| = |\beta_\infty(P,\Sigma)$$

or $|M_\infty(P,\Sigma)|$ = next Suslin cardinal after $|\beta_\infty(P,\Sigma)|$ in $M_\infty(P,\Sigma)$