Steel 10/23

Based on:

1) Mouse Pairs and Suslin cardinals, Steel

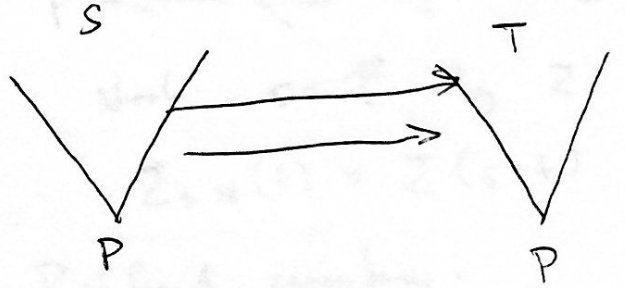
2) Suslin cardinals and mouse limits, Jackson-Sargsyan-Steel

3) Mouse pairs and Suslin cardinals in a Type 1 hierarchy, Steel

Handwritten notes, available on request

① Mouse pairs: (usually - and for this talk - background theory is AD^+ ($+AD_{\mathbb{R}}$))
pairs of the form
(P, Σ), w/ P ctbl transitive premouse (strategy or pure ext.)
and Σ an (ω_1, ω_1)-iteration strategy for P s.t. Σ has

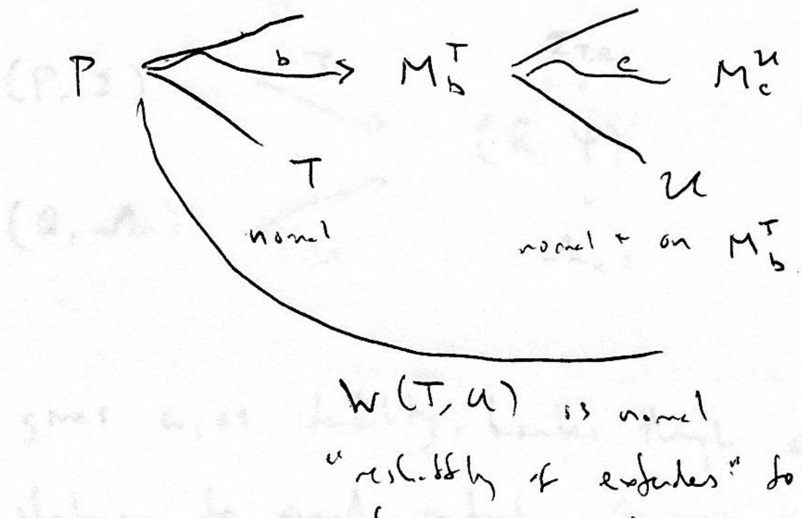
(1) strong hull condensation



$\Phi: S \to T$ a tree embedding, then T by Σ $\Rightarrow$ S is by Σ

"Σ condenses to itself by tree embeddings"

(2) normalizes well



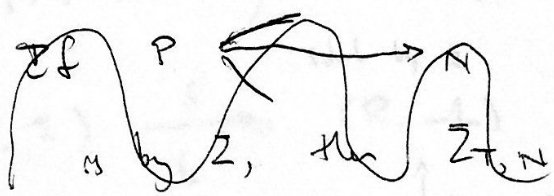
$W(T, U)$ is normal
"reslicability of extenders" to make a normal tree

T, U by $\Sigma \Rightarrow V(T, U)$ by Σ

- works for stacks of any length

(3) Internal Lift Consistency.

(4) (P, Σ) strategy pair $\Rightarrow$ (P, Σ) has pullback consistency: the fragments of P's iteration strategy agree w/ the external strategy Σ.



$P \Leftarrow\!\!\!\longleftarrow \longleftarrow \longleftarrow - - N$ by Σ

stack $s = \vec{T}$ by Σ

$\Sigma_{s,N}(t) = \Sigma(s^\frown t)$ is the tail strategy

Pullback consistency:

Pf s is by Σ w/ last model N,

$$\dot{\Sigma}^N \subseteq \Sigma_{s,N}.$$

- Can compare mouse pairs of the same type.

$$\begin{tikzcd}
(P, \Sigma) \arrow[dr, "T"] & & \\
& (R, \Psi) & \\
(Q, \Lambda) \arrow[ur, "U"'] & &
\end{tikzcd}$$

$\Sigma_{T,R} = (R, \Psi)$'s $\Psi = \Lambda_{U,R}$

T, U ctbl by Σ, Λ resp.

(AD^+ gives $\omega_1 + 1$ iterability, branches through ω_1-length trees.
Mouse strategies do avoid cardinality Aronszajn trees.)

In the comparison, at most 1 of the branches drops.

- Dodd-Jensen Theorem

$$(P,\Sigma) \le^* (Q,\Lambda) \quad \text{iff } P\text{-to-}R \text{ does not drop on } T$$

$\uparrow$

Mouse order (preorder by DJ Lemma)

- DJ implies minimality of iteration maps

stabl by Σ

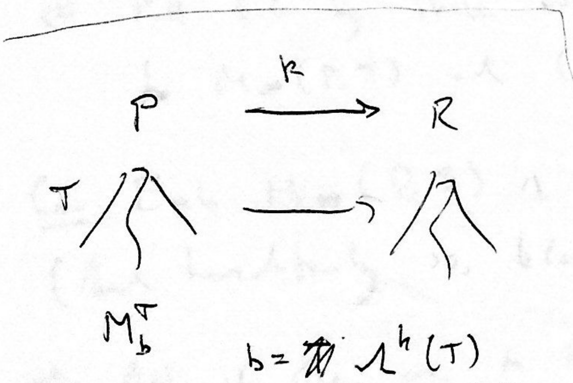
$$(P,\Sigma) \xrightarrow[i_s]{s} (R,\Lambda)$$

$\uparrow k : P \to R$ elementary in fine structural sense, and

(P,Σ) $\qquad \Lambda^k = \Sigma$

$\uparrow$

k-pullback of Λ:



$$\Lambda^k(T) = \Lambda(kT)$$

$$i_s(\eta) \le k(\eta) \ \forall \eta.$$

"Iteration maps are minimal".

If k is also a iteration ~~s~~ map, then $k = i_s$.

Cor $\exists$ at most 1 iteration ~~strategy~~ map from (P,Σ) to (R,Λ).

Mouse Limits: (P,Σ) a mouse pair,

$$F(P,\Sigma) = \{(Q,\Lambda) : (Q,\Lambda) \text{ is a non-dropping iterate of } (P,\Sigma) \text{ by a ctbl stack}\}$$

$$\begin{tikzcd} & (Q_1,\Lambda_1) \arrow[dr, "j_1"] & \\ (P,\Sigma) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & (R,\Psi) \\ & (Q_2,\Lambda_2) \arrow[ur, "j_2"'] & \end{tikzcd}$$

~~By~~ $j_1 \circ i_1 = j_2 \circ i_2$ by DJ

$\Rightarrow$ $M_\infty(P,\Sigma) = \text{dir lim } \mathcal{F}(P,\Sigma)$ makes sense

Then for $(Q,\Lambda) \in \mathcal{F}(P,\Sigma)$

$\pi_{(Q,\Lambda),\infty}$ is the map $(Q,\Lambda) \to (M_\infty(P,\Sigma), \_)$

Prop. ~~Also~~ $(P,\Sigma) \equiv^* (Q,\Lambda)$ iff
$M_\infty(P,\Sigma) = M_\infty(Q,\Lambda)$

$\Rightarrow$

$\Leftarrow$ Pull back by Skolem hulls of trees that get from (P,Σ) to $M_\infty(P,\Sigma)$ and (Q,Λ) to $M_\infty(Q,\Lambda)$.

Cor Each $M_\infty(P,\Sigma)$ is OD. (from its place in $\leq^*$)
(And hereditarily so, b/c P is wellordered structure)

$\Rightarrow$ All of them are in HOD

$\alpha \mapsto H_\alpha$ = Common $M_\infty(P,\Sigma)$ for (P,Σ) w/
$\leq^*$-rank $= \alpha$

is in HOD

($AD_{\mathbb{R}}$ $\Rightarrow$ $L[\alpha \mapsto H_\alpha] = \text{HOD}$.)

Suslin cardinals

• $A \subseteq \mathbb{R}$ is **κ-Suslin** iff $A = p[T]$ for T on $\omega \times \kappa$

In AD settings cannot prove nearly all sets of reals have a Suslin representation (essentially).

They give essential info about A

• κ is a **Suslin cardinal** iff $\exists A \subseteq \mathbb{R}$ A κ-Suslin but not λ-Suslin $\forall \lambda < \kappa$.

• $S(\kappa) = \{A \subseteq \mathbb{R} : A \text{ is } \kappa\text{-Suslin}\}$.

closed downward under $\leq_W$
under $\exists^{\mathbb{R}}$, $\bigcap_\omega$, $\bigcup_\omega$

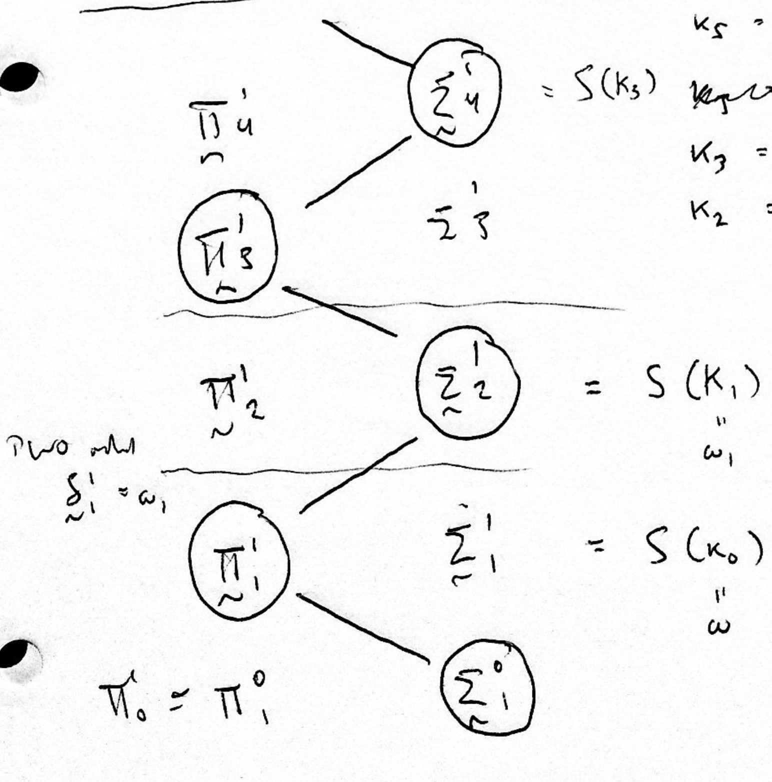
(Kechris) • Not closed under $\neg$ if κ Suslin

• Not closed under $\forall^{\mathbb{R}}$ in general

• $S(\kappa)$ or $\breve{S}(\kappa)$ has scale property.

• (AD^+) $\{\kappa : \kappa \text{ Suslin cardinal}\}$ is a closed (in order topology on ordinals) subset of Θ. (Woodin)

Suslin cardinals + norms ~~pw~~ limits in the projective hierarchy



$\kappa_5 = \delta^1_5 = (\kappa_4)^+$

$\kappa_4 = \delta_{Env_3} = \aleph_{\omega^\omega}$ cof $= \omega$

$\kappa_3 = \kappa_2^+ = \delta^1_3$

$\kappa_2 = \aleph_\omega = \delta_{Env_2}$ (Hjorth)

$Env_2 = Env(\Pi^1_2 \cup \Sigma^1_2) = \bigcup_n ⅁\, \omega n\text{-}\Pi^1_1$

$Env(\Pi^1_1 \cup \Sigma^1_1) = \bigcup_n \omega^n\text{-}\Pi^1_1$

[**Obs.** Suslins alternate btw cof ω and regular]

How far to get scales on non-Suslin pointclasses least witness

$\kappa_2 = o\,(\cancel{M_1 \mid \delta_1^{M_1}, \Sigma_{M_1}})\,(M_\infty(M_1 \mid \delta_1^{M_1}, \Sigma_{M_1}))$

$\aleph_2$

$\exists\, \delta_1^{M_1}$

P_1

$\kappa_1 =$ If $\beta_\infty = \pi_{P_1,\infty}$(least strong # to δ_1, $C(M_1)$)

then $|\beta_\infty| = \omega_1$

This is an instance of a general pattern of characterizing c.f. ω, then regular cardinals.

$\kappa_4 = o\,(M_\infty(M_3 \mid \delta_1^{M_3}, \Sigma_{M_3}))$

P_2

$\kappa_3 = |\beta_\infty|$, where $\beta_\infty =$ (least strong to κ_4 in M_∞)

- Want to generalize this to higher levels of $L(\mathbb{R})$