Ralf 11/2

(*) versions of MM

- Forcing over ZFC + LC models vs ZF + AD models
- Martin's Maximum^{++}:

If $\mathbb{P}$ is a stationary set preserving forcing, and

$\mathcal{D} = \{D_i : i < \omega_1\}$ is a collection of dense sets, and

$\{\tau_i : i < \omega_1\}$ is a collection of names for stat subsets of ω_1, i.e. $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\tau_i \subset \omega_1$" is stat,"

then $\exists$ filter $g \in V$ s.t.

$g \cap D_i \neq \emptyset$, $\forall i < \omega_1$

$\tau_i^g = \{\xi : \exists p \in g \; p \Vdash \xi \in \tau_i\}$ is stationary,

$\forall i < \omega_1$.

Exercise.

MM^{++} is equivalent to:

For all $\mathbb{P}$ stat set preserving,

for all models $\mathcal{M} = (M, \vec{R})$

and for all Σ_1-formulae

at most $\aleph_1$-many relations, functions

$M \; \mathcal{L}_{\in, NS_{\omega_1}}$,

if $V^{\mathbb{P}} \models \phi(\mathcal{M})$, then $\exists \; \bar{\mathcal{M}} \prec \mathcal{M}$ in V s.t. $\phi(\bar{\mathcal{M}})$.

Let's formulate a strengthening of MM^{++}.

Def. Let ϕ be a Σ_1 formula in $\mathcal{L}_{\in, NS_{\omega_1}}$.

Let $\mathcal{M}$ be a model as above.

Say that $\phi(\mathcal{M})$ is <u>honestly consistent</u> iff for all universally Baire functions F: $\exists$ transitive model $A \in V^{Col(\omega, tc(\{\mathcal{M}\}))}$ which is F-closed and s.t.

$$A \models ZFC^- + \phi(\mathcal{M})$$

$$tc(\{\mathcal{M}\}) \subseteq A$$

$$NS_{\omega_1}^V = NS_{\omega_1}^A \cap V$$

- $MM^{*,++}$: MM^{++} equivalent w/ $\phi(\mathcal{M})$ honestly consistent replacing "$V^{\mathbb{P}} \models \phi(\mathcal{M})$"
- $MM^{*,++}_\lambda$: $|\mathcal{M}| = \lambda$

$\sim$ all dense sets are of size $\leq \lambda$.

- $\Gamma - MM^{*,++}_\lambda$

$\Gamma \subseteq \Gamma^\infty$

For all $A \in \Gamma$, for all models...

$\mathcal{L}_{\in, NS_{\omega_1}, \dot{A}}$

... universally

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~~$MM^{(*),++}_\lambda \Rightarrow MM^{(*)}$~~

- $MM^{++}_\lambda \Rightarrow MM^{++}(\lambda)$ is MM^{++} restricted to $|\mathbb{P}| \leq \lambda$
- **Def** $(\Gamma-)BMM^{(*),++} \equiv (\Gamma-)MM^{(*),++}_{\aleph_1}$

<u>Theorem</u> (NS_{ω_1} saturated + V is closed under $X \mapsto M_\omega^\#(X)$).

Equivalent are

1) $P(\mathbb{R}) \cap L(\mathbb{R}) - BMM^{*,++}$

2) $(*)$ ($\equiv$ Woodin's $\mathbb{P}_{max}$ axiom)