1. Viale – Simon's Tutorial 18 XII '23

Definition $(*)_{uB}$ is $P(\omega_1) \subseteq L(uB)[G]$ for some $G \in V$ which is P-max generic over $L(uB)$.

$(*)_{uB} \Rightarrow L(uB)[G] \models NS$ is saturated, but it's not clear if $(*)_{uB} \Rightarrow V \models NS$ is saturated.

In my places $T_{\forall\exists}$ should be replaced with $T_{\forall \vee \exists} = T_{Bool\,\exists}$

Theorem
Assume LC + Sealing. Then:

(i) $(*)_{uB}$ + NS is saturated $\rightarrow$ ~~Th(V~~

$\Rightarrow$ ~~Th(V~~ $\underbrace{Th(H_{\omega_2}, NS, A : A \in uB)}_{(**)_{uB}}$ is model complete.

(ii) $(**)_{uB} \rightarrow (*)_{uB}$.

Fact
Let $\in_{\Delta_0, uB} = \{ \in_{\Delta_0} \cup \{R_\phi :$ quantifiers of ϕ range in $L(uB)$ and $\phi(\vec{y}) \longleftrightarrow \vec{y}$ is a tuple of elements of $2^\omega \} \}$

We take R_ϕ s.t. $\forall \vec{y} \left(\phi(\vec{y}) \longrightarrow \bigwedge_{i=1}^{n} y_i \in 2^\omega \right)$
if $\forall x$ occurs in in $\phi(\vec{y})$, x must range in $L(uB)$.

Our axioms are $\forall \vec{x}\ R_\phi(\vec{x}) \longleftrightarrow \phi(\vec{x})$.

Theorem
TFAE for $T \supseteq ZFC_{\in_{\Delta_0}, uB, NS} + LC + \text{Sealing}$, and ψ – a Π_2-sentence of $\in_{\Delta_0, uB, NS}$:

(i) ψ is $R_{\forall\exists}$ is consistent $\forall R \supseteq T$
(ii) $(H_{\omega_2}, \in_{\Delta_0}, NS, R_\phi : \phi \in \mathcal{Q}) \models \psi$ in models of $T + MM^{++} + T_{\forall\exists} + ZFC_{\in_{\Delta_0}, uB, NS} + \text{Sealing}$
(iii) ψ is in the model companion of T

Definition Given τ-theory T, $\tau_{\forall\exists} = \{\psi \in \mathcal{I} : \psi$ is boolean combination of universal sentences$\}$

$$T_{\forall\exists} = \{\psi \in \tau_{\forall\exists} : T \models \psi\}$$

Given $A \subseteq \tau \times 2$, $\tau_A = \tau \cup \{R_\phi : \langle \phi, 0\rangle \in A\}$
$\cup \{f_\psi : \langle \psi, 1\rangle \in A\}$

Assume T is τ-theory s.t. $T \vdash \forall \vec{x}\, \exists! y\, \psi(\vec{x}, y)$ for all $\langle \psi, 1\rangle \in A$.

Then every model of T has a unique expansion to a τ_A-model of $T + \forall \vec{x}\,(\phi(\vec{x}) \leftrightarrow R_\phi(\vec{x}))$
$+ \forall \vec{y}\, \forall x\, (\psi(\vec{x}, y) \leftrightarrow f_\psi(\vec{x}) = y)$

Definition Given τ-theories T, S, they are <u>absolute cotheories</u> if $T_{\forall\exists} = S_{\forall\exists}$.

S is the <u>absolute model companion</u> if T, S are absolute cotheories and S is model complete.

Example $\underbrace{Th(\mathbb{Q}, +, \cdot, 0, 1)_{\forall\vee\exists}}_{S_0} \neq \underbrace{Th(\mathbb{C}, +, \times, 0, 1)_{\forall\vee\exists}}_{T_0}$ as signaled by $\exists x\,(x^2+1=0)$

but T_0 is the model companion of S_0.

Definition Given τ-theory T, $SPEC_{AMC}(T) = \{A \subseteq \tau \times 2 : T + Ax^i_\phi, \langle \phi, i\rangle \in A\}$ has AMC$\}$ ~~has the AMC~~.

~~Theorem Assume $A \in AMC$ $ZFC_{\in \Delta_0}$~~

~~$A \in$~~

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Theorem Assume $A \in SPEC_{AMC}(T \quad)$ $T \supseteq ZFC_{\in\Delta_0}$

(i) If CH remains Σ_2 in $\underbrace{T + Ax_\phi^i : \langle \phi, i \rangle \in A}_{T + T_{2,A}}$. Then $CH \notin AMC(T + T_{2,A})$

(ii) The same for Π_2 a ψ which is Π_2 in $\in\Delta_0 \cup \{\omega_1\}$
and $ZFC + \psi \to 2^{\aleph_0} = \aleph_2$). + ω_1 is first uncountable cardinal

ψ is not in $(T + T_{2,A})_{\forall\vee\exists}$ $\to$ $\neg\psi \notin$ ~~ACM~~ $AMC(T + T_{2,A})$.