Matteo 12/12 

**Theorem** Assume Sealing. TFAE: (over ZFC + PCWC + $\exists$ class of supercompacts ~~...~~ + $NS_{\omega_1}$ precipitous)

1. $(V, \in) \models (*)_{UB}$

2. $Th(H_{\omega_2}, \in_{\Delta_0}, NS_{\omega_1}, A : A \in \Gamma^\infty)$ is model complete (and is the model companion of $Th(V[G], \in_{\Delta_0}, NS_{\omega_1}^{V[G]}, A^{V[G]} : A \in \Gamma^\infty)$ for any forcing $\mathbb{P} \in V$ and $G$ $V$-generic for $\mathbb{P}$.)

**Rmk** Stronger than Woodin's theorem, at least potentially, b/c not restricted to forcible statements.

**Def** (Sealing)
If $G \subseteq \mathbb{P} \in V$ is $V$-generic and $H$ is $V[G]$-generic for $\mathbb{Q} \in V[G]$, then

$$L(UB)^{V[G]} \to L(UB)^{V[G][H]}$$
$$A \mapsto A^{V[G][H]} \text{ for all } A \in UB^{V[G]}$$

and any injection of $Ind^{L(UB)^{V[G]}}$ into $Ind^{L(UB)^{V[G][H]}}$,
and $\Theta$ is regular and $AD_{\mathbb{R}}$ in $L(UB)$.

**Note** The proof needs
1) every $A \subseteq 2^\omega$ definable in $(H_{\omega_2} \cup UB, \in)$ is $UB$
2) the definitions made in $(H_{\omega_1} \cup UB, \in)$ are forcing invariant
3) if $\{A_n : n < \omega\} \subseteq UB$ then $\prod_n A_n$ is still $UB$.

**Note** Parenthetical part of Theorem 2 follows since for $\psi$ $\Pi_2$-sentence in the signature and $T = Th(V, \in_{\Delta_0}, NS_{\omega_1}, A : A \in \Gamma^\infty)$
$\psi \in Th(H_{\omega_2}, \in_{\Delta_0}, NS_{\omega_1}, A : A \in \Gamma^\infty)$ iff $\psi$ is universally consistent over $T_{\forall\vee\exists}$ ($\psi + T_{\forall\vee\exists}$ is consistent)

$(*)_{UB}$: If $A \subseteq \omega_1$ is s.t. $\omega_1^{L[A]} = \omega_1$, then

$$G_A = \{(M,a) \in \mathbb{P}_{max} : \exists \mathcal{J} = \{J_{\alpha\beta} : \alpha \le \beta \le \omega_1^V\} \text{ iteration of } M \text{ s.t. } j_{0\omega_1}(a) = A\}$$

is $L(UB)$-generic for $\mathbb{P}_{max}$, and

$P(\omega_1)^{L(UB)[G_A]} = P(\omega_1)$

(and in $V$, there are large cardinals.)

<u>Note</u>: Assume Sealing.

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① $\Rightarrow$ ②.

~~Given an $\exists$ formula $\exists x \psi(x, A)$, $A \subseteq \omega_1$ WLOG, we must find a universal formula $\forall x\ \theta$~~

Given an existential formula $\exists x\, \psi(x,y)$, we must find a universal formula $\forall z\ \underbrace{\theta_\psi(z,y)}_{\text{quantifier-free}}$ s.t. $T \vdash \forall y\, (\exists x\, \psi(x,y) \leftrightarrow \forall z\, \theta_\psi(z,y))$

Since $V \models (*)_{UB} + NS_{\omega_1}$ precipitous,

let $\gamma$ be inaccessible and pick $M \prec V_\gamma$ ctbl.

Let $\bar{M}$ be the collapse, then $\bar{M}$ is iterable, and

for $a \in P(\omega_1)^{\bar{M}}$ w/ $\omega_1^{L[a]} = \omega_1^{\bar{M}}$,

$(\bar{M}, a)$ is in $\mathbb{P}_{max}$, and if $J_M : \bar{M} \to H_\gamma$ is the uncollapse,

$j_M(a) = A$, $\omega_1^{L[A]} = \omega_1^V$

$J_M : UB^M \hookrightarrow UB^V$.

$$\begin{tikzcd} M \arrow[r] \arrow[d, shift left, "\pi_M"] & M_1 = \{f(M \cap \omega_1) : f \in M\} \arrow[r] \arrow[d, shift left] & M_2 = \{f(M_1 \cap \omega_1) : f \in M_1\} \arrow[r] & \ \\ \bar{M} \arrow[r] \arrow[u, shift left, "J_M"] & \{[f]_{G^{\bar{M}}} : f \in \bar{M},\ f : \omega_1^{\bar{M}} \to \bar{M}\} \arrow[r] \arrow[u, shift left, "f(M\cap\omega_1) \,\leftarrow\, {[f]_{G^{\bar{M}}}}"] & \{ \end{tikzcd}$$

$M_{\omega_1}$ & $H_\gamma$

$\uparrow\downarrow$

$\bar{M}_{\omega_1}$

$G_M = \{S \in M \mid M_{\omega_1} \in S\}$ is $M$-generic for $P(\omega_1)/NS_{\omega_1}$

$H_{\omega_2} \ldots \models \exists x \varphi(x, A)$

Let $D = \{\bar{M}$ iterable : any $NS_{\omega_1}$-correct iteration of $\bar{M}$ in its $J_{\omega, \omega_1}(\bar{M})$ has its $H_{\omega_2}$ ... in $V\}$

$\forall J$ : $J$ is $NS_{\omega_1}$-correct iteration of $\bar{M} \in D$

with $a \in \bar{M}$, $j_{0\omega_1}(a) = A$,

$\bar{M} \models \exists x \varphi(x, a)$.

· There are such $J$ that works, so "$\forall J$" trivial...