Matteo 12/11 ①

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$G_{\Delta_0} = \{R_\phi : \phi \text{ a } \Delta_0 \text{ property}\} \cup \mathrm{Rud} \cup$ Gödel operations

$ZF^-_{\Delta_0}$ = Replacement for quantifier-free formulas
Collection
All axioms except power set
} Base theory

$ZFC^-_{\Delta_0} = ZF^-_{\Delta_0}$ + Choice

Plus:

$\forall y_1 \dots y_n, z \, (R_{\forall x \in z \phi}(z, y_1 \dots y_n) \Longleftrightarrow \forall x \, (x \in z \rightarrow R_\phi(x, z, y_2 \dots y_n)))$
and similar axioms for Rud etc

Point is to have axioms for abs concepts
N.f a nominal axiomatization
Happy as long as Π_2 theory.

If $\psi(\bar{x})$ is $\Delta_1(T)$ and $t_i(\bar{x}) = y$ is also $\Delta_1(T)$ for all $1 \leq i \leq n$

then $\psi(t_1(\bar{x}) \dots t_n(\bar{x}))$ is $\Delta_1(T)$

If $\psi(\bar{z})$ is $\Delta_1(T)$, $T \supseteq ZF^-_{\Delta_0}$ and $\theta(\bar{x})$ is defined by transfinite recursion over $\psi(\bar{x})$, then $\theta(\bar{x})$ is $\Delta_1(T)$.

$A \subseteq (2^\omega)^n$, $A \in J^\infty$

NS_{ω_1}

Def Given a τ-theory T, M is T-e.c. if $\forall N \supseteq M$ FT $M \prec_1 N$
and there is one such N.

If M is T-e.c. and Π_2-complete, then $M \models \varphi$ ②

for any Π_2 φ s.t. $\varphi + T_{\forall\vee\exists}$ is consistent

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$\{\varphi : \varphi$ is boolean combination of Π_2-sentences$\}$

$\{\varphi : M \models \varphi$ for all T-e.c. $M\}_{\forall\exists} = \{\varphi : \varphi + T_{\forall\vee\exists}$ is consistent, φ is $\Pi_2\}$

There are complete theories S s.t. the class of S-e.c. models is not elementary.

Def S has T as its model companion if M is S-e.c. $\iff M \models T$

Theorem S has a model companion T iff T is $T_{\forall\exists}$ and T ~~is existential~~ proves that every universal φ is T-equivalent to an existential:

For $\forall \vec{x}\, \varphi(\vec{x}, \vec{z})$ w/ φ quantifier-free, there is $\theta_\varphi(\vec{y}, \vec{z})$ w/ θ quantifier-free s.t.

$\forall \vec{z}\ (\forall \vec{x}\, \varphi(\vec{x}, \vec{z}) \leftrightarrow \exists \vec{y}\ \theta_\varphi(\vec{y}, \vec{z})) \in T$.

Theorem (Shoenfield)

If $\mathbb{P}$ is any forcing, $(H_{\omega_1}, \in_{\Delta_0}) \prec_1 (H_{\omega_1}, \in_{\Delta_0})^{V^{\mathbb{P}}}$

Theorem (Woodin)

PCWC $\Rightarrow$ " " $\prec (H_{\omega_1}, \in_{\Delta_0})^{V^{\mathbb{P}}}$

So $(H_{\omega_1}, \in_{\Delta_0})$ is essentially existentially closed...

Q Is $Th(H_{\omega_1}, \in_{\Delta_0})$ model complete?

Answer: No!

Projective sets measure logical complexity of the theory.

<u>Lemma</u> ~~Suppose $A \subseteq \mathcal{P}(\bigcup_n \mathcal{P}((2^\omega)^n))$~~

Assume $\forall X \in \mathcal{A}\ \exists n\ X \subseteq \mathcal{P}((2^\omega)^n)$, and if $X, Y \in \mathcal{A}$ $X \cap Y \in \mathcal{A}$ and $X^c \in \mathcal{A}$ and $\pi_i[X] \in \mathcal{A}$.

~~Theorem~~ Then $Th(H_{\omega_1}, \in_{\Delta_0}, A : A \in \mathcal{A})$ is model complete

<u>Corollary</u> If $\mathcal{A} \subseteq \Gamma^\infty$ and satisfies assumptions of Lemma, $Th(H_{\omega_1}, \in_{\Delta_0}, A : A \in \mathcal{A})$ is model complete and gen abs in V.

Want to do same for H_{ω_2}.

<u>Theorem</u>

PCWC. Let $\mathbb{P}$ be any forcing. Then $Th_{\forall\exists}(V, \in_{\Delta_0}, NS, A : A \in \Gamma^\infty)$ is $Th_{\forall\exists}(V^{\mathbb{P}}, \in_{\Delta_0}, NS^{V^{\mathbb{P}}}, A^{V^{\mathbb{P}}} : A \in \Gamma^\infty)$.

<u>Remark</u>

- $ZFC \vdash \exists S(\neg NS(S) \wedge \neg NS(\omega_1 \setminus S))$.
- Need LCs bc:

  $(\omega_1^V = \omega_1^L)$ is Δ_1 expressible but not gen abs

<u>Proof</u>

Let φ be ~~Σ_1~~ Π_1, and assume $V \models \varphi$. V-generic $V[G] \models \neg\varphi$

Let $G \subseteq \mathbb{P}$ V-generic and $H \subseteq \tau_\delta^{\omega_1}$ $(= \mathbb{Q}_{<\delta})$ such that $G \in V[H]$

Let H_0 be $V[G]$-generic st $V[G][H_0] \models NS$ precipitous

and $V[G][H_0]$ is SSP extension of $V[G]$

Let $\gamma < \delta$ be st $V_\gamma[G][H_0] \models ZFC$

In $V[H]$, $V_\gamma[G][H_0]$ is ideable b/c nodal segment of $V[G][H_0]$, which satisfies $\neg\psi$.

Let J be an NS-correct iteration of $V_\gamma[G][H_0]$, $J \in U H(V, H)$.

$V[G][H_0] \models \neg\psi$:

("Ideable" w/ A-ideal for all $A \in (\Gamma^\infty)^V$.)

$\Rightarrow J_{0,\omega_1^{V[H]}}(M_0) \models \neg\psi$

$$\underset{M_\delta}{\overset{\shortparallel}{}}$$

$$\underset{\neg\psi}{\underset{\shortparallel}{(M_\delta, \in_\Delta, A^{M_\delta} : A \in \Gamma^\infty, NS^{M_\delta})}} \sqsubseteq \underset{\psi}{\underset{\shortparallel}{\overset{UH(V,H)}{(\cancel{V[H]}, \in_{\Delta_0}, NS^{V[H]}, A^{V[H]} : A \in \Gamma^\infty)}}}$$

Contradiction

$\square$