The supervenience of syntax on semantics in the foundational context

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- Do syntactic features supervene or in some sense can be read off semantic ones?
- How to recover syntax from semantic information?

Foundations of mathematics

- The question whether syntactic features supervene or in some sense can be recovered from semantic ones, can be posed in the foundations of mathematics context.
- In particular we ask the question, whether and under what conditions a given *model class $\mathcal{K}$* has a natural or implicit logic; or even a natural syntax.
- A model class is a class of structures of the same similarity type, but possibly of different cardinalities, that is closed under isomorphism.

This is part of the formalism freeness project, in which we studied the degree of entanglement of canonical mathematical structures with various canonical formal theories.

The idea here is that small changes in syntax (or more generally any perturbation in the framework) can induce massive effects, while on the other hand large perturbations in the framework, such as a change of logic, can have no effect on the relevant object.

For example, zero-one laws* for finite structures are sensitive to signature in the sense that relational structures satisfy the beautiful zero-one law, but once one adds function symbols to the language, even the simple sentence $\exists x(f(x) = x)$ has limit probability $1/e$. (Theorem is due to Fagin.)

On the other hand many mathematical objects are indifferent to the underlying logic with which they are formalised, in the sense that one can change the underlying logic to another very different logic, but the object remains the same. Gödel's L for example, is built over first order logic; but there is a very large class of logics that one can use to build the constructible sets, and still L stays the same.

(*The probability of a random relational structure on the domain 1, . . . , n satisfying a given first order formula tends to either 0 or 1 as n tends to infinity)

This instability is interesting as mathematics as it is generally practiced is impervious to these framework perturbations. The natural language of the mathematician contains typically a mixture of logical idioms.

Instead of tracking syntactic variation in this way, here we consider the following situation: a class of structures equipped with no syntax, and there is no logic in the backgroud. (Though a model class does come with a *vocabulary* or *similarity type*.)

- Model classes are a fundamental object of study in everyday mathematics.
- Some model classes: groups, linear orders, free groups, fields, ordered fields, equivalence relations, vector spaces. (In logic: models of arithmetic, well-orders, transitive models of ZFC, etc.)
- We consider model classes in order to mod out the "noise" coming from individual structures. (Individual structures may have accidental properties leading to "accidental syntax".)

- In this talk I will explore this question from the point of view of some model theoretic developments, both recent and not.
- One question we are asking is whether syntax is part of the definition of a logic at all.

An early example of recovering syntax from semantics: extracting the proposition from a truth table

TRACTATUS LOGICO-PHILOSOPHICUS

5.101 The truth-functions of every number of elementary propositions can be written in a schema of the following kind:

(TTTT)(p, q)	Tautology	(if p then p, and if q then q) $[p \supset p \,.\, q \supset q]$
(FTTT)(p, q)	in words:	Not both p and q. $[\sim(p \,.\, q)]$
(TFTT)(p, q)	„ „	If q then p. $[q \supset p]$
(TTFT)(p, q)	„ „	If p then q. $[p \supset q]$
(TTTF)(p, q)	„ „	p or q. $[p \vee q]$
(FFTT)(p, q)	„ „	Not q. $[\sim q]$
(FTFT)(p, q)	„ „	Not p. $[\sim p]$
(FTTF)(p, q)	„ „	p or q, but not both. $[p \,.\, \sim q : \vee : q \,.\, \sim p]$
(TFFT)(p, q)	„ „	If p, then q; and if q, then p. $[p \equiv q]$
(TFTF)(p, q)	„ „	p
(TTFF)(p, q)	„ „	q
(FFFT)(p, q)	„ „	Neither p nor q. $[\sim p \,.\, \sim q$ or $p \mid q]$
(FFTF)(p, q)	„ „	p and not q. $[p \,.\, \sim q]$
(FTFF)(p, q)	„ „	q and not p. $[q \,.\, \sim p]$
(TFFF)(p, q)	„ „	p and q. $[p \,.\, q]$
(FFFF)(p, q)	Contradiction	(p and not p; and q and not q.) $[p \,.\, \sim p \,.\, q \,.\, \sim q]$

Those truth-possibilities of its truth-arguments, which verify the proposition, I shall call its *truth-grounds*.

Disjunctive Normal Form in propositional logic

p	q	r	f
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

$$f = (\neg p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r)$$

First answer to the question, when does a model class have a syntax?

- When it is definable in a logic with syntax.[1]
- Example: the class of linear orders is definable in first order logic.

[1] Not every logic has a nontrivial syntax, e.g. Shelah's L^1_κ.

- $\mathcal{K} = Mod(\phi)$ for ϕ a sentence of some logic $\mathcal{L}^*$.
- Van Heijenoort: "The proposition [in the abstract logic approach JK] remains unanalyzed, being reduced to a mere truth value."[2]
- Here the proposition is reduced to (identified with) its class of models.

[2] "Logic as calculus, logic as language," [van Heijenoort, 1967].

Second answer to the question, when does a model class have a syntax?

- When the model class *behaves* as if it had a syntax in the sense of answer 1, that is, if some consequences of having a syntax can be detected.
- If a model class $\mathcal{K}$ does not have those consequences, it probably does not arise from a logic with syntax.[3]

[3] "Arise" means $\mathcal{K} = Mod(\phi)$, where ϕ is in the logic.

Part 1: Spectra

In order to find out whether a model class is definable in some interesting logic, it is useful to investigate the cardinalities of models in the class. One might think that mere cardinalities are too rough a measure of any form of logicality but this is, in fact, surprisingly informative. This approach leads us to the concept of a spectrum:[4]

Definition

If $\mathcal{K}$ is a model class, the *spectrum* of $\mathcal{K}$ is the class $\mathrm{sp}(\mathcal{K})$ of cardinalities of models in $\mathcal{K}$ i.e.

$$\mathrm{sp}(\mathcal{K}) = \{|M| : \mathcal{M} \in \mathcal{K}\}.$$

[4] See also [Sagi, 2018].

- Depending on $\mathcal{K}$, the spectrum can be a singleton, an interval of cardinals, an initial (or final) segment of the class of all cardinals, or something more complicated, such as the class of all limit cardinals, or all limit cardinals of cofinality ω.[5]
- Even the patterns of finite numbers in spectra of first order sentences is highly interesting.[6] However, we are here concerned with infinite cardinals in a spectrum.

[5] In fact any set of cardinals can be the spectrum of a logic via a generalized quantifier.

[6] Related to open questions in computational complexity such as P=NP?

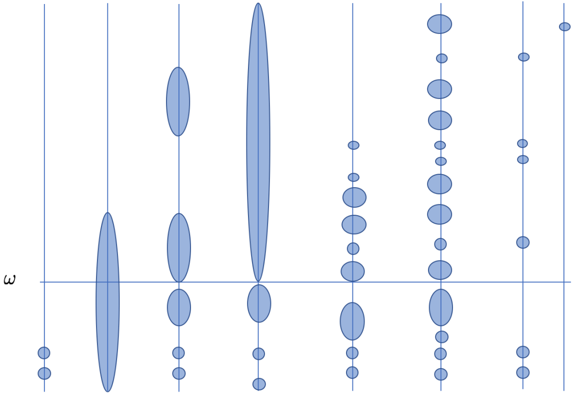

Figure: Spectra of some model classes.

- The property of a logic which reflects regularity patterns in its spectra is captured by the Löwenheim-Skolem Theorem.
- The spectrum gives indirect information regarding the possibility that the model class is definable in some (nontrivial) logic.
- Roughly speaking, if the logic has a strong Löwenheim-Skolem property, then the spectra of definable model classes reflect this.
- If every sentence in the logic which has an infinite model has also a countably infinite model, the most famous case of a Löwenheim-Skolem property, then every spectrum with an infinite cardinal in it has also $\aleph_0$ in it.

$\mathrm{LS}(C, D)$

Definition

Suppose C and D are classes of cardinal numbers. A logic L^* satisfies the *Löwenheim-Skolem Property* $\mathrm{LS}(C, D)$ if every sentence in L^* which has a model of some cardinality in C has a model of some cardinality in D.

Skolem proved that first order logic satisfies $\mathrm{LS}([\aleph_0, \infty), \{\aleph_0\})$: countable first order theories have countably infinite models if they have infinite models at all.

A simple observation

If a logic satisfies $\mathrm{LS}(C, D)$, there are consequences for the spectra of definable model classes. Suppose $\mathcal{K}$ is definable in a logic with $\mathrm{LS}(C, D)$. Then we can make the following conclusions: If there is $\mathcal{M} \in \mathcal{K}$ with $|M| \in C$, then there is $\mathcal{N} \in \mathcal{K}$ with $|N| \in D$.

On the other hand, if $\mathcal{K}$ contains a model of cardinality κ but no models of cardinality λ, then $\mathcal{K}$ cannot be definable in a logic with $\mathrm{LS}(C, D)$ such that $\kappa \in C$ and $\lambda \in D$. *The point is that by looking at the spectrum of $\mathcal{K}$ we can make inferences about its definability in different logics.*

Thus, if we are given a model class $\mathcal{K}$ but no logic in which it would be definable (apart from the trivial $L(Q_{\mathcal{K}})$, see below), and we can discern **regular** patterns in $\mathrm{sp}(\mathcal{K})$, we may take it as an indirect indication (albeit not a proof) that $\mathcal{K}$ is definable in a logic with $\mathrm{LS}(C, D)$ for some C and D explaining the found patterns.

On the other hand, if $\mathrm{sp}(\mathcal{K})$ is **irregular**, we may take it as an indication that no such logic can be found. (Incidentally, the spectra for second order logic can be very interesting, as the Hanf and Löwenheim numbers for second order logic are in the range of supercompact cardinals, if such exist.)

Thus $\mathrm{sp}(\mathcal{K})$ gives implicit information about the possibility of finding a syntax for $\mathcal{K}$.

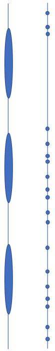

Figure: Regular and irregular spectrum.

Necessary, but not sufficient

A counterexample:

$\mathcal{K} = \{(\alpha + \alpha, <) : \alpha \in Ord\}$ has a nice spectrum but it is not definable in $L_{\infty,\infty}$.[7] However, it is definable using the game quantifier $\forall x_0 \exists x_1 \forall x_2 \exists x_3 \ldots \bigwedge_n \phi(x_0, \ldots, x_n)$.

[7] [Malitz, 1971].

Spectra for second order logic

Any class of successors of regular cardinals can be the spectrum of a second order sentence.

This follows from Easton's theorem[8] and from the fact that second order logic can detect whether $2^\kappa = \kappa^+$ or $= \kappa^{++}$ for cardinals below the size of the domain.

Can then write a sentence which has models exactly when $2^\kappa = \kappa^+$. The sentence has a unary predicate P and a binary predicate symbol $<$. It says that $<$ is a a well-order of the domain of type $|P|^+$, with the restriction of $<$ ordering P in the type of a regular cardinal, and that there is a function from the domain to subsets of P with every subset of P in its range.

[8] Suppose D is any class of successors of regular cardinals. Easton's Theorem says that there is such a forcing extension which forces $2^\kappa = \kappa^+$ for $\kappa^+ \in D$ and $2^\kappa = \kappa^{++}$ for regular κ with $\kappa^+ \notin D$.

Restrictions on spectra of second order sentences

Theorem

If a second order spectrum contains a measurable cardinal κ, it contains a stationary set of cardinals below κ.

Proof.

Suppose a measurable cardinal κ, with a normal[9] ultrafilter U, is in the spectrum of a second order sentence ϕ. Thus there is a model $\mathfrak{A}$ of ϕ of cardinality κ. W.l.o.g., $A = \kappa$. Let $i : V \to M$ be an elementary embedding of V into a transitive class M such that κ is the critical point of i and $M^\kappa \subseteq M$. In particular, $\mathfrak{A} \in M$. In M the sentence ϕ has a model, namely $\mathfrak{A}$ which is smaller than $i(\kappa)$. It follows that the set of $\lambda < \kappa$ such that ϕ has a model of cardinality λ is stationary. A subtle point: Why does $\mathfrak{A}$ still satisfy ϕ in M? This is because M is closed under κ-sequences, i.e. second order truth is preserved in the passage from $\mathfrak{A}$ to M. □

Corollary

A measurable cardinal cannot be the smallest element in a second order spectrum.

[9] I.e. if $f : A \to \kappa$ is pressing down and $A \in U$, then there is $B \subseteq A$ such that $B \in U$ and $f \restriction B$ is constant.

Theorem (Magidor)

Suppose κ is supercompact.[10] *No non-empty second order spectrum can consist only of cardinals $\geq \kappa$.*

Proof.

Suppose C is the spectrum with an element $\lambda \geq \kappa$. Suppose C is the spectrum of the second order sentence ϕ. Thus ϕ has a model $\mathcal{A}$ of size λ. By Magidor's Löwenheim-Skolem Theorem for second order logic $\mathcal{A}$ has a second order elementary submodel $\mathcal{B}$ of cardinality $\mu < \kappa$. But then $\mu \in C$, contrary to the assumption. □

[10] I.e. for any $\lambda > \kappa$ there is an elementary embedding $i : V \to M$ such that κ is the critical point and $M^\lambda \subseteq M$.

We have said. . .

While Easton's Theorem gives a lot of flexibility for second order spectra, there are restrictions imposed by large cardinals:

- Measurable cardinals in the spectrum have to have accumulation points.
- The spectra cannot live completely above a supercompact cardinal.

There are many other restrictions.

Structure!

- If ϕ is a second order sentence, recall that we defined

$$Spec(\phi) = \{|M| : M \models \phi\}.$$

- Since we can quantify over predicates, we may assume, w.l.o.g., that the vocabulary of ϕ is $\emptyset$. Hence the complement of a second order spectrum is also a spectrum. (Observed by various people including Durand et al.)
- For the finite parts of *first order* spectra this is a famous open problem ("Asser's Problem").

Second order spectra are not only closed under complement. . .

- The second order spectra form an infinite atomic Boolean algebra B. Such are all first order elementarily equivalent, so the complete theory of any one of them is decidable (Tarski).
- The Boolean algebra B can be obtained as follows: Define $\phi \sim \psi$ if $Spec(\phi) = Spec(\psi)$. B is isomorphic to the natural Boolean algebra obtained from the set of all $[\phi]$ where $\phi \in L^2$ has empty vocabulary. From spectra back to syntax!
- $-[\phi] = [\neg\phi], [\phi] \wedge [\psi] = [\phi \wedge \psi]$.

- As we noted, second order spectra can be complicated: For any $A \subseteq \omega$ there is a forcing extension in which $\{\aleph_n | n \in A\}$ is a second order spectrum.
- There is a second order ϕ such that if $0^\sharp$ exists, then $0^\sharp \leq_1 \{n < \omega : \aleph_n \in Spec(\phi)\}$.[11]
- Let $Spec^L(\phi)$ be the cardinals κ for which ϕ has a model of size κ in Gödel's L, in the sense of L.
- Assume $0^\sharp$. Some cardinal $\aleph_n$, $n > 0$, is in $Spec^L(\phi)$ iff every $\aleph_n$, $n > 0$, is. (The $\aleph_n$'s are indiscernables in L.)

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[11] Identify $0^\sharp$ with the set of Gödel number of formulas $\phi(v_1, ..., v_k)$ such that $L \models \phi(\aleph_1, ..., \aleph_k)$. The sentence ϕ says "the cardinality of the universe is a cardinal of the form $\aleph_n$, where n codes a pair (m, k) such that the m:th formula of set theory is satisfied in L by $\aleph_1, ..., \aleph_k$. Now, if $\phi(v_1, ..., v_k)$ is given, we map its Gödel number m one-one to the number n which codes m and k and we ask whether n is in $\{n < \omega : \aleph_n \in Spec(\phi)\}$.

Wolves?

The topic of second order spectra is mathematically rich. There are constraints on such spectra, and there is structure. If entanglement were malevolent, one would expect completely wild behavior instead of this mathematical richness.

Part 2: Logicality

Recall our answer to the question, when does a model class have a syntax:

When it is definable in a logic with syntax.

Every model class trivially gives rise to a logic

Note that every model class is definable in some logic because we can take the model class as a generalized quantifier in the sense of P. Lindström [Lindström, 1966], and then $\mathcal{K}$ is definable in $FO(Q_{\mathcal{K}})$.

Per Lindström

Suppose $\mathcal{K}$ is a model class with vocabulary L. For simplicity we assume $L = \{R\}$ where R is a binary predicate symbol. We can associate with $\mathcal{K}$ the generalized quantifier $Q_{\mathcal{K}}$ with the semantics

$$\mathcal{M} \models Q_{\mathcal{K}} xy\phi(x, y, \vec{a}) \iff$$
$$(M, \{(b, c) \in M^2 : \mathcal{M} \models \phi(b, c, \vec{a})\}) \in \mathcal{K}.$$

Now $\mathcal{K}$ is definable in $FO(Q_{\mathcal{K}})$ by the sentence $Q_{\mathcal{K}} xyR(x, y)$.

A minimal logic is one consisting of just atomic formulas. That is, for every n-ary predicate symbol R there should be a formula $\phi_R(x_1, \ldots, x_n)$ (usually denoted $R(x_1, \ldots, x_n)$) in the logic such that for all $a_1, \ldots, a_n \in M$:

$$\mathcal{M} \models \phi_R(a_1, \ldots, a_n) \iff (a_1, \ldots, a_n) \in R^{\mathcal{M}},$$

and for any two terms t and t' in the variables $x_1, \ldots, x_n$ there should be a formula $\psi_{t,t'}(x_1, \ldots, x_n)$ (usually denoted $t = t'$) in the logic such that for all $a_1, \ldots, a_n \in M$:

$$\mathcal{M} \models \psi_{t,t'}(a_1, \ldots, a_n) \iff t^{\mathcal{M}}(a_1, \ldots, a_n) = t'^{\mathcal{M}}(a_1, \ldots, a_n).$$

Let us denote this logic At. If $\mathcal{K}$ is a model class and we add $Q_{\mathcal{K}}$ to the logic At, we obtain the smallest logic $At(Q_{\mathcal{K}})$ in which $\mathcal{K}$ is definable, as for $\mathcal{M} = (M, R^{\mathcal{M}})$:

$$\mathcal{M} \in \mathcal{K} \iff \mathcal{M} \models Q_K xyR(x, y).$$

After obtaining the minimal logic *At* we can add the propositional operations. If we then add existential and universal quantifiers we obtain first order logic *FO*.

Since $\mathcal{K}$ is closed under isomorphism, $FO(Q_{\mathcal{K}})$ is a logic in the sense of Lindström:

Definition ([Lindström, 1969])

An abstract logic (or just a logic) is a pair $\mathcal{L} = \langle S, T \rangle$, where S is a set (non-empty) and T is a binary (class) predicate holding between models (i.e. domains equipped with relations) and elements of S. If the pair (A, ϕ) is in T and $A \cong B$, then also the pair (B, ϕ) is in T (isomorphism closure). The elements of S are called "$\mathcal{L} - sentences$" and T is called the "truth-predicate of $\mathcal{L}$."

Miettinen, Lindström, Krynicki, Jensen, Westerståhl

Oulu Summer School on Mathematical Logic 1974

What does it mean for an abstract logic to have a syntax?

- For an abstract logic $\langle S, T\rangle$ to have a syntax we need to say what having a syntax means.
- There is the trivial solution of simply listing the elements of S, the $\mathcal{L}$-"sentences".
- This is unsatisfactory. The syntax of e.g. English is not the list of all English sentences.

A better trivial solution

- Take the generalized quantifier $Q_{\mathcal{K}}$ for each $\mathcal{K} = Mod(\phi), \phi \in S$.
- The resulting logic $\mathcal{L}' = FO(\{Q_{\mathcal{K}} : \mathcal{K} = Mod(\phi), \phi \in S\})$ is equivalent to $\mathcal{L}$ in the sense that $\mathcal{L}$ and $\mathcal{L}'$ have the same definable model classes.[12]
- This is only a slight improvement from the previous "listing" solution.
- If a logic has a syntax there should be other consequences besides generating the definable model classes.

[12] Assuming $\mathcal{L}$ is closed under substitution in an exact sense.

- The natural idea is that the syntax consists of a small number of logical operations which generate S in a ‘recursive’ way.
- This is an apparently syntactic notion, but we can view the notion of “logical operation” semantically.

Logical operations can be viewed as model classes

Disjunction as a model class:

$$\mathcal{K}_{\vee} = \{(M, P_0, P_1, P) : P_0 \cup P_1 = P \subseteq M^n\}$$

Simpler: the class of unary models (i.e. the generalized quantifier consisting of models) (M, A), where $A \subseteq M$, A^M non-empty, and the class of (M, A), where A^M is equal to M. On the side of syntax these would correspond to the existential and the universal quantifiers.

In general

Logical operations (viewed semantically) map semantic values to semantic values. : Suppose M is a non-empty set. By the semantic value (on M) of a formula (of any logic) we mean the set of assignments into M that satisfy the formula.

A *(local) operation f on M* maps sequences $\langle A_\alpha : \alpha < \beta \rangle$ of sets $A_\alpha \subseteq M^{n_\alpha}$ to sets $f(\langle A_\alpha : \alpha < \beta \rangle) \subseteq M^n$.

Such a local operation is a *logical operation* if it is closed under permutations of M.

Logicality

This is related to the issue of the logicality of a model class as we say that the operation corresponding to the model class is logical to the degree that the model class is definable in a nice logic, i.e. a logic that is close to being first order with respect to its model theoretic properties.[13]

Permutation invariance on its own is a questionable guarantor of logicality, due to *overgeneration*. E.g. the cardinality of the underlying domain is classified as logical under the criterion.

[13] This was the position taken in "Logicality and model classes," [Kennedy and Väänänen, 2021].

So far we have said...

The model class $\mathcal{K}$ is trivially definable in the extension $FO(Q_{\mathcal{K}})$ of first order logic by the quantifier $Q_{\mathcal{K}}$ by the sentence $Q_{\mathcal{K}}xyR(x,y)$.

Conversely, every class of models definable in $FO(Q_{\mathcal{K}})$, or indeed in any abstract logic, is a model class i.e. is closed under isomorphisms.

An easy theorem

Theorem

If $\mathcal{K}$ is a class of models of the same vocabulary, then the following conditions are equivalent:

1. *$\mathcal{K}$ is closed under isomorphisms, i.e. $\mathcal{K}$ is a model class.*
2. *$\mathcal{K}$ is definable in some extension of first order logic by a generalized quantifier.*
3. *$\mathcal{K}$ is definable in some logic.*

Corollary

An operation is a logical operation (i.e. can be identified with a model class via a generalized quantifier) if and only if it can be expressed in some logic, i.e. if and only if the associated model class is definable in some logic.

A theorem of V. McGee improves "some logic" to a very specific logic, namely $L_{\infty\infty}$, but at a price: *we obtain a different definition for each cardinality separately*.[14]

[14] [McGee, 1996]

In order to state McGee's theorem we identify the property of *cardinality dependence*. For any cardinal λ let $\mathcal{K}_\lambda$ be the class of elements of the model class $\mathcal{K}$ with a domain of size λ.

Definition

A model class $\mathcal{K}$ is *cardinal dependently definable*, or *CD-definable*, in a logic L^*, or *cardinal dependently L^*-definable*, if $\mathcal{K}_\lambda$ is L^*-definable for every λ.

Cardinality Dependence

Note that a model class can be CD-definable even in first order logic without being definable in $L_{\infty\infty}$: Consider the class $\mathcal{K}_{\text{lim}}$ of models (M, P), where either $|M|$ is a limit cardinal and $P = \emptyset$ or else $|M|$ is a successor cardinal and $P \neq \emptyset$. We have now defined the class $\mathcal{K}_{\text{lim}}$ in a first order way in every cardinality, i.e. this class is CD-definable in first order logic.

But it is a consequence of the Löwenheim-Skolem Theorem[15] of $L_{\infty\infty}$ that this model class cannot be definable in it. (Suppose the class is definable by ϕ in $L_{\kappa^+\kappa^+}$. We can use the Löwenheim-Skolem Theorem to move from a model of $\phi \wedge \exists x P(x)$ of successor cardinality to a submodel of limit cardinality (or vice versa) and thereby violate the definition of $\mathcal{K}_{\text{lim}}$.)

[15] If a sentence of $L_{\kappa^+\kappa^+}$ has a model $\geq \mu^\kappa$ for some μ, it has a model of size μ^κ.

McGee's theorem

Theorem ([McGee, 1996])

If $\mathcal{K}$ is a class of models of the same vocabulary, then the following conditions are equivalent:

1. *$\mathcal{K}$ is closed under isomorphisms.*
2. *$\mathcal{K}$ is CD-definable in $L_{\infty\infty}$.*

A "generalization" of DNF

Proof sketch.

Let us fix an infinite cardinal λ. Let $\mathcal{M}_\alpha$, $\alpha < 2^\lambda$, be an enumeration of all elements of $\mathcal{K}$ with domain λ. Let $\theta_\alpha \in L_{\lambda^+\lambda^+}$ characterize up to isomorphism the model $\mathcal{M}_\alpha$.[16] The sentence $\bigvee_{\alpha<2^\lambda} \theta_\alpha$ defines the model class $\mathcal{K}_\lambda$. □

[16] θ_α is essentially the diagram of $\mathcal{M}_\alpha$.

- The fact that McGee's theorem depends on the cardinality of the models in the class means that the sentence of ϕ_λ of $L_{\infty\infty}$ defining the model class $\mathcal{K}_\lambda$ may very well depend on λ, as in the example $\mathcal{K}_{\text{lim}}$.
- When we ask whether the property of a model of belonging to the model class $\mathcal{K}_{\text{lim}}$ is logical or not, we essentially ask whether the property of a model (or more precisely $|M|$) of having a limit cardinality is logical or not.
- According to the Tarski-Sher criterion of permutation invariance[17] it is logical. On the other hand, it is fair to say that it is a mathematical property rather than a logical one.
- From the point of view of "Logicality and Model Classes" the logicality of membership in $\mathcal{K}_{\text{lim}}$ manifests in a rather low degree of logicality.

[17] The criterion classifies a notion as logical if it is invariant under all permutations of the relevant domain. See Tarski's "What is a logical constant?" See also [Sher, 2008].

First-orderism

- This leads us to ask whether $L_{\infty\infty}$ can be replaced by a "tamer" logic, one closer to being first order in its model-theoretic properties?
- If such were to exist, then even with the problem of dependence on the cardinality of the models in the class unsolved, the relevant logicality claim would be strengthened by virtue of its proximity to first order logic.
- In fact $L_{\infty\infty}$ can be replaced by a logic which is absolute and which has a strong Löwenheim-Skolem theorem, together with other desirable properties, though it is still the case that the definition is given relative to the size of the models in the class.

Theorem

If $\mathcal{K}$ is a class of models of the same vocabulary, then the following conditions are equivalent:

1. *$\mathcal{K}$ is closed under isomorphisms.*
2. *$\mathcal{K}$ is CD-definable in $\Delta(L_{\infty\omega})$.*[18][19]

The Δ-operation preserves properties like compactness, axiomatizability, Hanf and Löwenheim numbers. The Δ-operation "fills the gaps" left by explicit definability in the sense that if a model class is implicitly definable in the logic then it is explicitly definable in the Δ-extension. Essentially, when we consider $\Delta(L)$ rather than the logic L itself, we focus on what the logic becomes when some accidental weaknesses are removed.

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[18] See [Kennedy and Väänänen, 2021]. Theorem is due to Väänänen.

[19] A model class is said to be Δ-definable in a logic L if it is the class of reducts of models of a sentence of L, and also its complement is.

Back to having syntax

To say that a model class behaves as if it had a syntax, is to cash out "behavior" in terms of consequences:

- It has a nice spectrum.
- It has Löwenheim-Skolem type properties of some sort, e.g. holds for all $L_{\kappa\lambda}$ and all $L(Q_\alpha)$.
- It has completeness theorems of some sort, e.g. holds for $L_{\omega_1\omega}, L(Q_1), L(aa)$, and in a generalized sense for $L_{\kappa\lambda}$ $L(Q_\alpha)$ and even second order logic L^2 (by modifying the semantics).

Having a logic

Having Löwenheim-Skolem type properties of some sort, e.g. holding for all $L_{\kappa\lambda}$ and all $L(Q_\alpha)$, is a marker of logicality, for here we obtain indifference to cardinality.

More consequences of having a syntax

- It has compactness theorems of some sort. (Usually follows from completeness.)
- Proof theory of some sort. (Part of the proof of completeness.)
- Model theory of some sort. (Using compactness.)
- Set theoretical absoluteness of some sort. (As in first order logic, $L_{\infty\omega}$).

The trivial syntax gives none of these.

It is difficult to define a notion of syntax that would cover all the relevant cases but exclude the trivial solutions.

What we can infer if $\mathcal{K}$ has "strong" Löwenheim-Skolem properties

A candidate for a strong Löwenheim-Skolem property:

Definition ([Lindström, 1966])

We say that a model class $\mathcal{K}$ is *reducible* if for all $\mathfrak{A}$ the following hold:

1. $\mathfrak{A} \in \mathcal{K} \Rightarrow$ there is a club of countable subsets $B \subseteq A$ such that $\mathfrak{B} \in \mathcal{K}$, where $\mathfrak{B}$ is $\mathfrak{A} \restriction B$.
2. $\mathfrak{A} \notin \mathcal{K} \Rightarrow$ there is a club of countable $\mathfrak{B} \subseteq \mathfrak{A}$ such that $\mathfrak{B} \notin \mathcal{K}$.

Lindström proved that if $\mathcal{K}$ is reducible, then every model class definable in $L(Q_{\mathcal{K}})$ is reducible. This means that $\mathcal{K}$ is then definable in a logic $\mathcal{L}^*$ (namely $L(Q_{\mathcal{K}})$) with the property that for every $\phi \in \mathcal{L}^*$:

$$\mathfrak{A} \models \phi \iff \text{there is a club of countable subsets } B \subseteq A \text{ such that } \mathfrak{B} \models \phi.$$

(Proof by induction on the complexity of ϕ.)

This means that if $\phi \in \mathcal{L}^*$ has a model, it has a countable submodel, and there are enough countable submodels to form a club.

Some reducible logics: $FO, L_{\omega_1,\omega}, L(Q_0)$, any absolute logic.

Replacing $\aleph_0$ by any regular κ we have the same for submodels of size κ for $L(Q_{\alpha+1})$ $(\aleph_\alpha = \kappa)$, $L_{\kappa^+\omega}$.

- Knowledge about the reducibility of $\mathcal{K}$ helps us to find a reducible logic in which it is definable, namely $L(Q_{\mathcal{K}})$.
- This logic does not have a particularly informative syntax but reducibility in itself is an indication of possible definability in a nice syntax.

Improving Lindström's theorem

- Define

$$\mathcal{M} \models \text{aa}\ s\ \phi$$

$$\iff \text{there is a club of countable } s \text{ such that } \mathcal{M} \models \phi(s).$$

- aa is the almost all quantifier.
- $L(\text{aa})$ is stationary logic. It is axiomatizable (Shelah, Barwise, Makkai, Kaufmann) and countably compact.

Theorem

If the model class $\mathcal{K}$ is reducible, it is definable in $L_{(2^\omega)^+\omega}(\text{aa})$.

This is better than Lindström's result for arbitrary $\mathcal{K}$, because $L_{(2^\omega)^+\omega}(\text{aa})$ has a more informative syntax.

DNF again

Proof.
If $\mathcal{N}$ is a countable model of the vocabulary of $\mathcal{K}$, let $\theta_{\mathcal{N}} \in L_{\omega_1\omega}$ characterize it up to isomorphism (proved by Scott, assuming a countable vocabulary). For s a unary predicate, set $\theta^{(s)}_{\mathcal{N}}$ be the relativization of $\theta_{\mathcal{N}}$ to the predicate s. Let ϕ be the sentence

$$\text{aa } s \bigvee \{\theta^{(s)}_{\mathcal{N}} : N \subseteq \omega, \mathcal{N} \in \mathcal{K}\}.$$

Now ϕ defines $\mathcal{K}$.[20] □

[20] If $\mathcal{M} \in \mathcal{K}$, there is a club of countable $\mathcal{N} \subseteq \mathcal{M}$ such that $\mathcal{N} \in \mathcal{K}$. Hence $\mathcal{M} \models \phi$. If $\mathcal{M} \notin \mathcal{K}$, then there is a club E of countable $\mathcal{N} \subseteq \mathcal{M}$ such that $\mathcal{N} \notin \mathcal{K}$. If $\mathcal{M} \models \phi$, there is a club E' of countable $s \subseteq M$ for which $M \cap s$ satisfies $\bigvee\{\theta^{(s)}_{\mathcal{N}} : N \subseteq \omega, \mathcal{N} \in \mathcal{K}\}$. Let $\mathcal{N} \subseteq \mathcal{M}$ such that $N \in E \cap E'$. There is $\mathcal{N}' \in \mathcal{K}$ such that $\mathcal{N}' \cong \mathcal{N}$. Hence $\mathcal{N} \in \mathcal{K}$, a contradiction with $N \in E$.

Part 3: ZFC as syntax

One may always view model classes as set-theoretic objects:

$\mathfrak{A} \in \mathcal{K} \iff \Psi(\mathfrak{A}, a)$, for Ψ a first order formula in the language of set theory and a is a parameter.

Note that if $\mathcal{K} = Mod(\phi)$ for ϕ in some logic $\mathcal{L}^*$, then

$$\mathfrak{A} \in \mathcal{K} \iff \mathfrak{A} \models_{\mathcal{L}^*} \phi,$$

so here $\Psi(x, y)$ is the formula $x \models_{\mathcal{L}^*} y$ and the parameter a is ϕ.

Internal vs external definability, i.e. is Ψ a syntax of $\mathcal{K}$?

Is this “external” set-theoretic definability a syntax of $\mathcal{K}$? Does it help us to find such? Note that set-theoretic definability adds $\in$ to the vocabulary of $\mathcal{L}^*$. So, overkill.

Symbiosis[21] relates what happens inside a model to how the model sits in the set-theoretic universe V.

[21] [Väänänen, 1979]

"Inside" vs "Outside"

If in $\mathcal{K}$ we actually have ϕ in a logic $\mathcal{L}^*$ defining the class, then we have

$$\mathfrak{A} \in \mathcal{K} \iff \mathfrak{A} \models \phi \iff \Psi(\mathfrak{A}, a).$$

ϕ looks at $\mathfrak{A}$ from the inside,

Ψ looks at $\mathfrak{A}$ from the outside.

- Symbiosis was invented in order to explain the non-absoluteness of second order logic; to answer the question, what keeps second order logic from being absolute?
- An exact answer: the nonabsoluteness of the relation $R(x, y)$ for "x is the power set of y" is the reason why second order logic is non-absolute.
- Once we adopt R-absoluteness, that is to say once we hold the power set operation fixed, second order logic becomes absolute.[22]
- On the other hand, second order logic "sees" the predicate R and can talk about it and everything else that is R-absolute, via its definable model classes.[23]

[22] The absoluteness of second order logic means here that the satisfaction relation is absolute for transitive models of set theory when the power-set operation R is respected (i.e. absolute). Of course SOL is otherwise famously nonabsolute.

[23] The model class $\mathcal{K}_{\mathcal{R}}$ "associated with R" is Δ-definable in L^2 (generalized definable).

In other words, the power set operation is symbiotic with second order logic. (As one would expect.)

A model class has the "inside" definability property of the left column if and only if it has the "outside" definability property of the right column. The properties on the right can be viewed as degree of absoluteness. (The traditional absoluteness in set theory is the same as Δ_1-definability.)

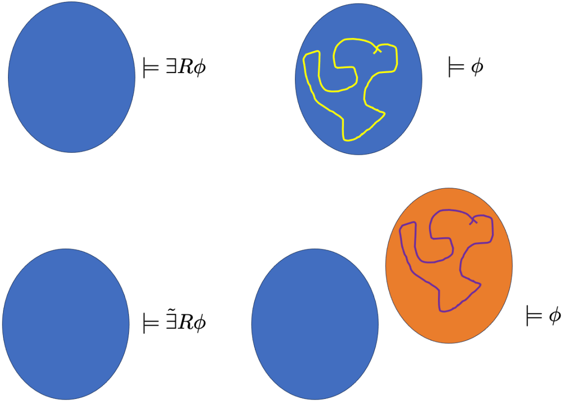
Inside	Outside
Sort logic[24]	First order logic
Second order logic	$\Delta_1(P)$ in the Levy-hierarchy
First order logic with the Härtig-quantifier	$\Delta_1(Cd)$ in the Levy-hierarchy
Infinitary logic L_{HYP}	Δ_1^{KP} in the Levy-hierarchy
First order logic	$\Delta_1^{KPU^-}$ in the Levy-hierarchy

$$\mathcal{M} \models \phi \iff \Phi(\mathcal{M})$$

[24] In Sort Logic one is allowed to "guess" predicates outside the domain of the model.

The complexity of the set-theoretical definition Ψ is an indicator of the difficulty of finding a syntax for the model class.

Sort Logic: "guessing" predicates outside the domain of the model.



Definition

A logic L is *symbiotic* with a predicate P of set theory if:
On the one hand:

- The predicate "$\phi \in L$" is $\Sigma_1(P)$. (generalized r.e.)
- The predicate "$M \models_L \phi$" is $\Delta_1(P)$. (generalized rec.)
- A certain canonical model class $\mathcal{K}_P$ associated with P is definable in the Δ-extension of the logic L.[25]

[25] A model class is said to be definable in the Δ-extension of a logic L if it is the class of reducts of models of a sentence of L, and also its complement is.

Generic absoluteness

Note that second order logic over the natural numbers, i.e. over the model $(\omega, +, \times)$, **is** absolute under (set) forcing if we assume a proper class of Woodin cardinals.

Part 4: Games!

We raised the possibility of extracting information about the definability of a model class from raw semantic data, e.g. the spectrum of the class.

Let's consider another type of data, namely a game.

If a logic $\mathcal{L}$ is given, we can define the following equivalence relation:

$$\mathfrak{A} \equiv_{\mathcal{L}} \mathfrak{B} \iff \forall\phi \in \mathcal{L}(\mathfrak{A} \models \phi \leftrightarrow \mathfrak{B} \models \phi).$$

Now assume we are given an arbitrary equivalence relation $\equiv^*$ on structures, closed under isomorphism.

Question: Does $\equiv^*$ arise from a logic?

I.e. is there are logic $\mathcal{L}^*$ such that $\equiv^*$ is the same as $\equiv_{\mathcal{L}^*}$?

Another trivial solution

Let S be the collection of all $\equiv^*$-classes. Consider $L^* = \langle S, T \rangle$, where

$$T(\mathfrak{A}, \phi) \iff \mathfrak{A} \in \phi, \text{ for } \phi \in S.$$

Note that here ϕ is an $\equiv^*$-class $[\mathfrak{A}_0]$ of L^*-structures, so really we have

$$T(\mathfrak{A}, \phi) \iff \mathfrak{A} \equiv^* \mathfrak{A}_0.$$

- The problem is that a priori S (i.e. the $\equiv^*$-classes) is a proper class, while in the Lindström definition, S must be a set.
- But if $\equiv^*$ satisfies a kind of Löwenheim-Skolem Theorem, namely there is κ such that for all $\mathfrak{A}$ there is $\mathfrak{B}$ of size $\leq \kappa$ in the $\equiv^*$-class of $\mathfrak{A}$, then it is enough to take S to be the (now) set of $[\mathfrak{A}]$ with universe $\subseteq \kappa$.
- Now $|S| \leq 2^\kappa$.
- Thus the Löwenheim-Skolem Theorem helps us to find a reasonable logic L^* such that $\equiv^*$ is the same as $\equiv_{\mathcal{L}^*}$. (Proof easy.)

Tools from the finite realm

Two finite graphs G and H are isomorphic iff for every finite graph F, we have $|hom(F, G)| = |hom(F, H)|$. (Note that $|hom(F, G)|$, $|hom(F, H)|$ are finite). (L. Lovasz.)

Call the "left profile of G" the infinite vector consisting of all homomorphism counts $|hom(F, G)|$, as F varies over all finite graphs.

Restrict F to a class C of graphs and consider the following equivalence relation: two graphs G and H are equivalent if and only if they have the same left profile restricted to C.

Dvořák showed that if C is the class of all graphs of treewidth at most k, then the associated equivalence relation is elementary equivalence in $(k-1)$-variable FO-logic with counting quantifiers.[26]

[26] "On Recognizing Graphs by Numbers of Homomorphisms," Wiley 2009. See also "On the Expressive Power of Homomorphism Counts", Atserias, Kolaitis, Wei-Lin Wu.

Shelah's L^1_κ

- Shelah's logic L^1_κ ([Shelah, 2012]) has interpolation[27] and satisfies a kind of Löwenheim-Skolem theorem.
- L^1_κ is the only strong logic with interpolation besides $\mathcal{L}_{\omega_1\omega}$. Moreover, it's maximal w.r.t. the Löwenheim-Skolem theorem and an undefinability of well-order condition (a weak form of compactness).
- Cf. Lindström's Theorem, a semantic characterization of first order logic, i.e. another way of reading syntax off semantics.
- The "logic" has no syntax but it has a criterion of elementary equivalence given by a game,[28] *whence it is a logic according to Shelah.*

[27] i.e. disjoint model classes, definable with extra predicates, can be separated by a definable model class

[28] I.e. if player 2 has a winning strategy in the game then the structures are equivalent according to the eq. rel. of the game.

Knots in natural language

- Is L^1_κ a logic? Does L^1_κ have a reasonable syntax?
- M. Džamonja, Siiri Kivimäki, B. Veličković, A. Villaveces, and J. Väänänen are developing various ways toward finding a syntax.
- The game paradigm in logic is fundamental:
    - The Ehrenfeucht-Fraïssé game providing a criterion for elementary equivalence.
    - The semantic game giving a criterion for truth in a model.
    - The model existence game which gives a criterion for consistency.

In sum

In my monograph [Kennedy, 2020] I examined the semantic point of view, and attempted to say something exact about the entanglement of semantically presented canonical (mathematical) structures with canonical formal languages. This material is a continuation of that work, but now starting simply with raw semantic data and **no** syntax.

Thank you!

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