How first order is first order logic?

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Introduction

Fundamental to the practice of logic is the dogma regarding the first order/second order logic distinction, namely that it is *ironclad*. Was it always so? The emergence of the set theoretic paradigm is an interesting test case. Early workers in foundations generally used higher order systems in the form of type theory; but then higher order systems were gradually abandoned in favour of first order set theory—a transition that was completed, more or less, by the 1930s.[1]

[1] According to Hodges the transition in Tarski's early work, at least, from simple type theory to informal set theory, was in place by then. As Hodges puts it: "The deductive theories in question (such as RCF) are formulated in simple type theory; by 1935 the axioms for RCF is regarded as a definition within set theory." [12], p. 118. See Ewald's [7] for the emergence of first order logic.

Gödel [*1933o] describes the higher order "provenance" of (first order) set theory—*the fact that set theory lends itself to being viewed in a natural way as a higher order system*—as follows:

> *It may seem as if another solution were afforded by the system of axioms for the theory of aggregates, as presented by Zermelo, Fraenkel and von Neumann; but it turns out that this system of axioms is nothing else but a natural generalization of the theory of types, or rather, it is what becomes of the theory of types if certain superfluous restrictions are removed.*[2]

[2] [9], p. 45. A similar point is made in the second author's [31]: "First-order set theory is merely the result of extending second order logic to transfinitely high types."

Set theory, then, has a double nature—logically speaking.

As for logic in general, there is some tension in what the phrase "first order," as in "first order logic," really means. It is clear what it means for a model class K to be first order definable.[3] It means that there is a first order formula ϕ such that for all models M

$$M \in K \iff M \models \phi. \tag{1}$$

On the other hand, K being merely a model class means, from the point of view of set theory, that there is a first order formula $\Phi(x)$, perhaps with set parameters, such that for all models M

$$M \in K \iff \Phi(M). \tag{2}$$

[3] By a model class we mean a class of models, closed under isomorphisms, of a fixed vocabulary.

A helpful metaphor: "inside" vs "outside"

While (1) holds for a very restricted collection of model classes only, namely the first order definable ones, (2) holds for *all* model classes. Still both (1) and (2) seem to be based on first order logic.

Obviously first order logic is playing a different role in (1) and (2): in (1) first order logic "views" the model M in some sense from the *inside*, while in (2) the perspective taken is from the *outside*, how M sits in the universe of sets.

This simple observation suggests that the concept of a logic being first order is not only about whether the variables range over the elements of a given domain, or over sets of elements, or over sets of sets of elements, and so on, it is also about the *context*. In (1) the first order variables of the defining formula ϕ range over elements of M, while in (2) the first order variables range over the universe of set theory V,[4] which contains sets generated by unbounded, even transfinite, iterations of the power set operation.

This means that higher order quantification (over set-size domains) is in a clear sense allowed in (first order) set theory.

[4]with the appropriate caveats

Of course, set theory is a *theory* and second order logic is a *logic*, at least that is the common understanding. We argue in [16][5] that if one cares to view set theory as a logic then set theory turns out to be a stronger logic than second order logic.

This is perhaps as it should be, given that the latter restricts the domain of quantifiable objects to those generated by (at most) a *single* iteration of the power set operation, while set theory allows for arbitrary iterations of the power set operation.

[5] "How first order is first order logic," to appear, *The Oxford Handbook of Philosophy of Logic*.

An elementary observation about the first order/second order distinction being context-dependent

Consider the structure

$$\mathcal{M} = (\mathbb{R}, +, \times, \mathbb{N}, <, 0, 1).$$

First order quantification over this structure involves quantification over the real numbers. Via their binary representation, definable in this structure, every real number corresponds canonically to a subset of $\mathbb{N}$. Thus when we quantify in a first order way over the real numbers we are implicitly quantifying in a second order way over natural numbers, because we can identify a real number with a subset of $\mathbb{N}$. Thus first order quantification over the reals can be viewed as second order quantification over the naturals.

The presence or non-presence of $\mathbb{N}$ as part of the structure $\mathcal{M}$ decides whether first order quantification over the model is truly first order or implicitly second order over an infinite substructure.[6]

From the point of view of $\mathbb{N}$ the quantification is second order, from the point of view of $\mathbb{R}$ it is first order.

[6]The first order theory of $\mathcal{M}$ is extraordinarily complex as it encodes the entire second order theory of $(\mathbb{N}, <, 0, 1)$, known to be non-computable in the extreme. This should be contrasted with the fact that the arithmetic of the reals alone is decidable [27]. The point is that the decidability concerns the structure $(\mathbb{R}, +, \times, <, 0, 1)$, in which $\mathbb{N}$ is not a part of the structure (it is not even a *definable* subset).

Quine made a related point in *Philosophy of logic* [24] when he suggested that using second order predicate symbols as schematic letters masks the set theoretic content of second order logic; **one should rather include the membership relation in the given signature.**

Another ambiguity in the notion of "first order" is due to the fact that there is a whole spectrum of logics which extend first order logic in the sense first orderness appears in (1) but are sublogics of first order logic in the sense first orderness appears in (2).

In fact, every (abstract) logic is first order from the point of view of set theory.

An *abstract logic* is given by two predicates of set theory, namely the set (or class) of formulas and the truth predicate, where the latter is a predicate of set theory holding between structures and sentences of the logic. Such predicates are given in set theory by a first order formula.

Of course the first order formula involves the epsilon relation. Hence in **first order logic plus the epsilon relation** one can define every logic.

Speaking of logic + ϵ: Conversations between Tarski, Carnap and Quine at Harvard, 1940s

Conversations were devoted to the question whether set theory belonged to logic or not; more broadly the aim was to devise a physicalistic theory for science.

Tarski: "mathematics = logic + ϵ."[7]

[7] Mancosu, "Harvard 1940-1941: Tarski, Carnap and Quine on a finitistic language of mathematics for science," [21].

A very simple example of the definability of every logic in set theory:

Example

A typical non-first order property of a model is its **finiteness.** Let K be the class of **finite** models of some vocabulary L. It is a familiar consequence of the Compactness Theorem[8] that there is no **first order** sentence ϕ of any vocabulary L such that for all models M the equivalence (1) holds (for the class K). On the other hand, if $\Phi(x)$ is the familiar[9] set-theoretical formula which says that the set x is a finite model of vocabulary L, then all models M of vocabulary L satisfy (2).

First order logic cannot express finiteness from **inside**; first order set theory easily expresses finiteness from **outside**.

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[8] Every theory, which has no models, has a finite subtheory without models.

[9] There are many different definitions of finiteness in set theory, all equivalent if the Axiom of Choice is assumed. The most common definition says that there is no one-one function from x into a proper subset of x.

So what does "first order" mean, after all, if first order logic can appear in such different roles as (1) and (2)?

We suggest that the distinction between first order and higher order logics, such as second order logic, is somewhat context dependent.

From the philosophical or foundational point of view this complicates the picture of first order logic as a canonical logic.

Absoluteness

Barwise [3] pinned the canonicity of a logic to its *absoluteness*:

> *When is it reasonable for us, as outsiders looking on, to call [an abstract logic] L^* a "first order" logic? If the words "first order" have any intuitive content it is that the truth or falsity of $M \models^* \phi$ should depend only on ϕ and M, not on what subsets of M may or may not exist in [the logician's] model of his set theory T. In other words, the relation $\models^*$ should be absolute for models of T.*

The absoluteness of a logic means that its set theoretical definition is absolute in the usual set theoretical sense, i.e. persisting upwards and downwards across transitive models of (of a finite part of) ZFC.

More precisely a logic is said to be *absolute* [3] if the satisfaction predicate $M \models \phi$ is Δ_1 in the Levy hierarchy, and the property of being a formula of the logic is Σ_1 in the Levy hierarchy.[10]

[10]The Levy hierarchy is defined as follows: Σ_0-formulas of set theory are formulas in which all quantifiers are bounded i.e. of the form $\forall x \in y$, $\exists x \in y$. Π_0-formulas are the same as Σ_0-formulas. A formula is Σ_{n+1} if it is of the form $\exists x \phi$, where ϕ is Π_n. A formula is Π_{n+1} if it is of the form $\forall x \phi$, where ϕ is Σ_n. A property of sets is Δ_n if it can be defined both by a Σ_n-formula and a Π_n-formula. If the equivalence can be proved in the theory T, typically Kripke-Platek set theory KP or ZFC, the property is called Δ_n^T. The Kripke-Platek axioms *KP* consist of some elementary axioms plus the Σ_0-separation and Σ_0-collection schemas.

Essentially a logic L is absolute if the truth of a sentence in an L-structure depends only on the elements of the domain, not on what kind of subsets it has.

For Barwise this was exactly the mark of a canonical logic.

First order logic is of course absolute, but there are many extensions of first order logic that are absolute in Barwise's sense, and therefore, **in Barwise's sense, first order, or at least close to first order, at least from the semantic point of view.** Examples of absolute logics include $L(Q_0)$ [23], $L(Q_0^{MM})$ [20], $L_{\infty\omega}$ [13], and the logic $L_{\infty G}$ [10, 22] with the closed game-quantifier

$$\forall x_0 \exists x_1 \forall x_2 \exists x_3 \ldots \bigwedge \phi_n(x_0, \ldots, x_{2n+1}). \tag{3}$$

So if we take absoluteness to be a marker of canonicity, we will not single out first order logic.

Many other important logics fall into the class of absolute logics.

In order to talk about the absoluteness of a logic more exactly we need the concept of an abstract logic:

A *logic* (a.k.a. abstract logic)[11] is a pair $L^* = (\Sigma, T)$, where Σ is an arbitrary set (sometimes also a class) and T is a binary relation between members of Σ on the one hand and structures on the other.

Members of Σ are called L^*-sentences.

Classes of the form

$$\mathrm{Mod}(\phi) = \{\mathcal{M} : T(\phi, \mathcal{M})\},$$

where ϕ is an L^*-sentence, are called L^*-characterizable, or L^*-definable, classes.

[11] in the sense of [18]

Abstract logics are assumed to satisfy five axioms expressed in terms of L^*-characterizable classes, corresponding to being closed under isomorphism, conjunction, negation, permutation of symbols, and "free" expansions.[12]

To say that a logic is absolute is to say that its truth definition, considered as a binary predicate in the language of set theory, is absolute for transitive models of set theory (also membership in Σ).

[12]The free expansion to vocabulary L of a model class K of a smaller vocabulary is the class of all expansions of elements of K to the vocabulary L.

The semantic point of view

A class $\mathcal{K}$ of models is said to be *definable* in a logic L^* if there is a sentence ϕ in L^* such that

$$\mathcal{K} = \mathrm{Mod}(\phi)$$

Van Heijenoort: "The proposition [in the abstract logic approach JK] remains unanalyzed, being reduced to a mere truth value."[13]

Here the proposition is reduced to (identified with) its class of models.

[13] van Heijenoort, "Logic as calculus, logic as language," [35].

Where do model classes come from?

Tarski [28] introduced the concept of an *elementary* class. This refers to the class of all models of a given first order sentence ϕ with vocabulary L. Thus elementary classes K satisfy:

1. The elements of K are models (i.e. structures).
2. All models in K have the same vocabulary L.
3. K is closed under isomorphisms.

Generalizing from this, we call any class K a *model class* if it satisfies the above three conditions. Elementary classes are

examples of model classes but there are many more. For example, the classes of all groups, all well-orders, all equivalence relations, all algebraically closed fields, all models of Peano arithmetic, all models of ZFC set theory, and the class of all models isomorphic to some $(V_\alpha, \in)$, where α is an ordinal, are model classes.

If ϕ is a sentence of any logic whatsoever, be it second order logic, logic with generalized quantifiers, or infinitary logic, the class of models of ϕ is a model class.

A model class doesn't necessarily come with a syntax or a logic. It has a **vocabulary**; but a vocabulary is **not** a syntax.

Consider the class of all groups $\mathcal{G}$. The vocabulary, or similarity type, for groups, consists of an associative binary operation, a unary operation (the inverse operation), and a zero-ary operation, namely the identity element: $\tau =< *, inv, 1 >$.

A syntax is a set of symbols subject to recursive formation rules, often attached to a logic.

An interesting fact about model classes is that every model class is definable in *some* logic, because we can take the model class as a generalized quantifier in the sense of Lindström [17].[14]

[14] Suppose $\mathcal{K}$ is a model class with vocabulary L. For simplicity we assume $L = \{R\}$ where R is a binary predicate symbol. We can associate with $\mathcal{K}$ the generalized quantifier $Q_{\mathcal{K}}$ in the sense of [17] with the semantics

$$\mathcal{M} \models Q_{\mathcal{K}} xy\phi(x, y, \vec{a}) \iff (M, \{(b, c) \in M^2 : \mathcal{M} \models \phi(b, c, \vec{a})\}) \in \mathcal{K}.$$

Now $\mathcal{K}$ is trivially definable in the extension $L_{\omega\omega}(Q_{\mathcal{K}})$ of first order logic by the quantifier $Q_{\mathcal{K}}$ by the sentence

$$Q_{\mathcal{K}} xyR(x, y).$$

Conversely, every class of models definable in $L_{\omega\omega}(Q_{\mathcal{K}})$, or indeed in any abstract logic, is a model class i.e. is closed under isomorphisms.

Talking about model classes is tantamount to talking about sentences in arbitrary logics.

Where we are. . .

We are trying to make the case that the distinction between first and second order logic is not as sharp as is generally thought.

Absoluteness is part of the story: absoluteness is a marker of canonicity and therefore absolute logics come close to being first order. (First order logic being the paradigm case of a canonical logic.)

Later I will offer other ways in which logics extending first order logic draw close to being first order. (Model-theoretic properties, internal categoricity, etc.)

In contrast to first order logic, second order logic is famously nonabsolute.

One can easily write a second order sentence Φ which is true in the ordered field $(\mathbb{R}, +, \cdot, 0, 1, <)$ of real numbers if and only if the Continuum Hypothesis CH holds,[15] and this equivalence is provable in ZFC.

But the CH is "forcing fragile," ergo so is its second order equivalent. A simpler example is uncountability. No absolute logic can express uncountability.[16]

[15] The CH says that every uncountable set of reals has the same cardinality as the set of reals itself. This is expressible in second order logic because we can quantify over all subsets of the domain, we can express countability and we can express being of the same cardinality as the entire domain. Let Φ be the second order sentence $\forall P(\Psi(P) \to \exists F \Theta(F, P))$, where $\Psi(P)$ says "P is uncountable" and $\Theta(F, P)$ says "F is a one-one function from elements of the domain into P." Then Φ holds in $(\mathbb{R}, +, \cdot, 0, 1, <)$ iff CH is true.

[16] See [3].

Symbiosis

The nonabsoluteness of second order logic must have to do with its set-theoretical content. **But what exactly is the set-theoretical content of second order logic?**

Symbiosis, introduced in [30], was designed exactly in order to bring the set theoretical content of a logic to the fore; to "expose the nature of the logic, to uncover the set-theoretical commitments of the logic, its content, its strength, even its reference."[17]

[17] See [14].

Symbiosis in my previous talk (part 1 of today's) was used to think about the problem of finding a natural syntax for a model class.

The question, how a vocabulary **grows** into a syntax and a logic. . . How a model class comes to carry a syntax and a logic.

Here I will use it to argue for the proposition that set theory is the strongest logic.

Precisely, in symbiosis one finds a set-theoretical predicate or operation P such that in any situation where P is absolute the logic $\mathcal{L}$ is, and (roughly) vice versa. **This means that one is able to detect, on the one hand, whether a logic "sees" the invariant content of a given set theoretic predicate; while on the other hand the absoluteness of the logic is pinned to the absoluteness of the predicate**—whence the name "symbiosis."

Second order logic is nonabsolute because it is *symbiotic* with the power set operation.[18]

That is to say, once we hold the power set operation fixed, second order logic becomes absolute. On the other hand, second order logic "sees" the power set operation and can talk about it and everything else that is "absolute relative to the power set operation," via its definable model classes.

[18] For Quine's view of the entanglement of second order logic with set theory see the section entitled "Set theory in sheep's clothing," [24], p. 66.

Of course it is not surprising that the nonabsoluteness of second order logic should be tied to the power set operation, somehow.

Ingredients in the definition of symbiosis

A predicate P is R-absolute if whenever we add sets to the universe or take sets away, without changing R, also P remains unchanged. For example, if $R(x)$ is the predicate "x is countable," then the predicates

- "x is uncountable,"
- "x is a countable ordinal,"
- "x is a countable set of singletons,"
- "$(A, <)$ is a linear order in which every initial segment is countable,"
- "G is a graph in which every node has uncountably many neighbours,"

are all R-absolute.

Technically. . .

Definition

Suppose R is a predicate (i.e. a first order formula) in the language $\{\epsilon\}$. A predicate P is *absolute w.r.t. R*, or R-absolute, if it is absolute with respect to transitive extensions preserving the predicate R, i.e. if the predicate is preserved by extensions of the universe as long as R itself is preserved, and no new elements are added to old elements (technically: extensions of transitive models); and the same is true of restrictions of the universe.

Technically this is the same as P being $\Delta_1(R)$ i.e. Δ_1 in the extended language $\{\in, R\}$ [8].

Δ-operation

- Suppose $\mathcal{K}$ is a model class of vocabulary τ.
- $\mathcal{K}$ is $\Sigma(L^*)$-definable if there is a bigger vocabulary τ' and a sentence ϕ of L^* in the vocabulary τ' such that $\mathcal{K}$ is the class of reducts of models of ϕ to τ.
- $\mathcal{K}$ is $\Delta(L^*)$-definable if both $\mathcal{K}$ and its complement are $\Sigma(L^*)$-definable.

Δ-operation

The Δ-operation preserves properties like compactness, axiomatizability, Hanf and Löwenheim numbers; it "fills the gaps" left by explicit definability in the sense that if a model class is "implicitly" definable in the logic then it is explicitly definable in the Δ-extension.

For example, $L(Q_0)$ cannot say that an equivalence relation has infinitely many equivalence classes,[19] although "morally" it should be able to do so, whereas $\Delta(L(Q_0))$ can say it easily.

[19] The proof of this is an easy application of the method of Ehrenfeucht-Fraïssé games.

“Plus ϵ...”

Essentially, when we consider $\Delta(L)$ rather than the logic L itself, we focus on what the logic becomes when some accidental weaknesses are removed.

Maybe $L(Q_0)$ was defined as it was because the generalized quantifier Q_0 is appealing in its simplicity. But it turns out that $\Delta(L(Q_0))$ is an infinitary logic, viz. the logic L_{HYP}, the smallest admissible fragment of $L_{\omega_1\omega}$ [4]. For a proper treatment of $L(Q_0)$ we have to consider the entire $\Delta(L(Q_0))$, otherwise our investigation may be centred around some accidental properties of $L(Q_0)$ with no general interest.

We now define the notion of symbiosis:

Definition

An n-ary predicate R and a logic $\mathcal{L}^*$ are *symbiotic* if the following conditions are satisfied:

1. Every $\mathcal{L}^*$-definable model class is absolute w.r.t. R.
2. Every model class which is absolute w.r.t. R is $\Delta(\mathcal{L}^*)$-definable.

The most blatant example of symbiosis is that already mentioned, between second order logic and the binary predicate "x is the power-set of y". Another is the symbiosis between the Härtig (or equicardinality) quantifier and the predicate $\mathrm{Cd}(x)$ i.e. "x is a cardinal number".

KPU^- is the theory KP with urelements minus the Axiom of Infinity. The Kripke-Platek axioms KP consist of some elementary axioms plus Σ_0-separation and Σ_1-collection schemas. L_{HYP} is the closure of first order logic under recursive conjunctions and disjunctions. Sort logic is defined below.

Inside: $\mathcal{M} \models \phi$	Outside: $\Phi(\mathcal{M})$	Reference
Sort logic	First order logic	[32]
Δ(Second order logic)	Δ_1(Pw) in the Levy-hierarchy	[30]
Δ(First order logic with the Härtig-quantifier)	Δ_1(Cd) in the Levy-hierarchy	[30]
Δ(First order logic with the game quantifier)	Δ_1 in the Levy-hierarchy	[6]
Infinitary logic L_{HYP}	Δ_1^{KP} in the Levy-hierarchy	[5]
First order logic	$\Delta_1^{KPU^-}$ in the Levy-hierarchy	[1]

In terms of the inside/outside metaphor, here the model theoretic definition of the class corresponds to the inside view, whereas the set theoretic definition of the class corresponds to viewing the class from the outside.

Set theory (expressed in a first order language) provides a first order way to quantify in any given structure over not only elements of the structure but also over subsets (second order logic!), sets of subsets (third order logic!), etc. **The intuition here is that (first order) set theory is a very strong, indeed the strongest logic.**

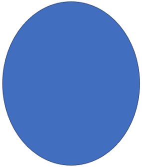
Sort logic

We want to understand the symbiotic relationship between sort logic and first order set theory.

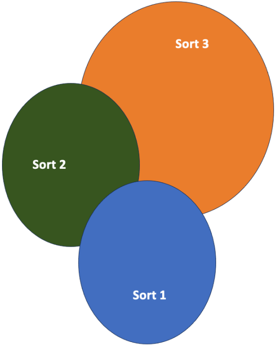
Recall that a relational structure, i.e. a model, has a domain and relations, functions and constants on that domain. A modification is a *many-sorted structure* [11] in which there are several domains and relations, functions and constants on those domains or between the domains.

A good example is a vector space, where there is a domain for scalars and a domain for vectors. A vector multiplied by a scalar is again a vector. To be able to talk about many-sorted structures in logic one adopts variables of different *sorts*, one sort for each domain. Thus in a language for vector spaces there is a sort for scalars and a sort for vectors. In other words, every individual variable has a sort attached to it and it is supposed to range over elements of the domain of that sort. Thus in a language for vector spaces there are variables for scalars and variables for vectors.

A model in one-sorted logic



A model in many-sorted logic



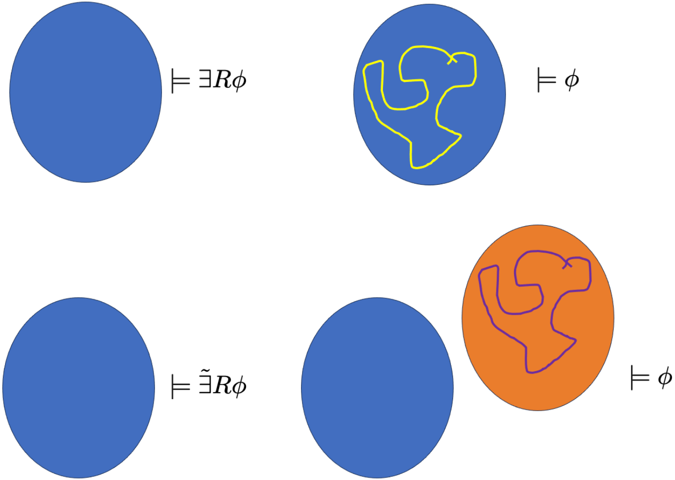
Sort logic [32] arises when we are allowed to quantify over a variable of a *new* sort i.e. a sort not present in our vocabulary.

Semantically this means that we claim there is a new domain that can be added as a new sort into our model and the expanded model satisfies what we want to say.

For example, we may want to say of a group that it is the multiplicative group of a field. We have to say that there is a zero-element outside the group so that the group together with the new element and a new addition form a field.

- In **first order** many-sorted logic there are individual variables for each sort.
- In **second order** many-sorted logic, i.e. **sort logic** we have also unary[20] relation variables $R_0, R_1, \ldots$. Each R_i ranges over subsets of individuals of some sort m_i.
- In a model which already has a domain of sort m_i the quantifier $\exists R_i \phi$ is defined as in second order logic.
- In a model which does not yet have a domain of sort m_i the quantifier $\tilde{\exists} R_i \phi$ is defined and means that one can add a domain to the model, and call it the domain of elements of sort m_i, and there is a subset of the new domain which satisfies in the new model ϕ.

[20] for simplicity, here.


$\models \exists R\phi$
$\models \phi$
$\models \tilde{\exists} R\phi$
$\models \phi$

- Sort logic allows us to guess a new domain and a subset of the new domain, while second order logic only allows us to guess only subsets of the current domain.
- For example, sort logic allows us to guess a model of set theory where our current model is an element. This is possible by first guessing a new domain and a subset that codes a binary predicate on the new domain. Then we state that the new binary predicate has to be well-founded and satisfy some axioms of ZFC, then, up to isomorphism, the new domain and the new predicate are a transitive model of a part of ZFC. Finally, we can say that the transitive model has an element which is isomorphic (second order condition!) to our current model. **Now we can look at our current model in the new model as if we were "outside" the current model.**
- Sort logic creates a bridge between the "inside" and the "outside" view.

About the symbiosis between sort logic and first order set theory:[21] In this case the symbiosis means that every model class definable by a sentence of sort logic is (a fortiori) definable in first order set theory, and, conversely, any model class definable in set theory by a first order formula is definable in sort logic. **This means that sort logic is the strongest logic; and thus, by symbiotic correspondence, set theory is too.**

Another way to say this: if you view set theory as a logic, then the logic that it "is" is sort logic.

[21] [32]

Symbiosis has applications beyond is its ability to calibrate the set-theoretic content of a logic.

Symbiosis can be used for:

1. The non-absoluteness of a logic can be delineated in terms of predicates of set theory. This may be useful for a better understanding of, for example, the behavior of the logic in forcing extensions.
2. One can relate Löwenheim-Skolem type model theoretic properties of logics with reflection or large cardinal properties of cardinals in set theory [2]. An early example of this is the fact that the smallest κ for which second order logic satisfies the Löwenheim-Skolem-Tarski Theorem at κ (i.e. for every second order sentence ϕ every model has an elementary submodel of cardinality $< \kappa$ in which ϕ is true) is exactly the same as the smallest supercompact cardinal [19].
3. On can relate the complexity of the decision problem of a logic[22] with set-theoretic definability criteria. An example of this is the result that the decision problem of second order logic is the complete Π_2-definable set of integers [29].

[22] The decision problem of a logic is the set of Gödel numbers of the valid sentences of the logic.

In conclusion, symbiosis lays down a bridge between the interior (1) and the exterior (2) view of a logic. In both perspectives first order logic is in a central role. From the **interior** point of view it is the weakest logic; from the **exterior** point of view it is the strongest.

Other ways to complicate the first order/second order distinction

Lindström's theorem characterizes first order logic in terms of certain canonical model theoretic properties.[23]

But some strong logics come very close to being first order by virtue of these properties, i.e. the logic satisfies a Compactness Theorem and a Downward Löwenheim-Skolem theorem in the same spirit as first order logic.

[23] Lindström's theorem states that first order logic is, up to equivalence of logics, the only logic closed under some elementary operations and satisfying the Compactness Theorem as well as the Downward Löwenheim-Skolem theorem (every sentence which has a model has a countable model).

Example (Cofinality logic)

[26] Consider the generalized quantifier

$$M \models Q^{\mathrm{cof}}_{\omega} xy\phi(x, y, \vec{a}) \iff$$
$$\{(b, c) \in M^2 : M \models \phi(b, c, \vec{a})\} \text{ is a linear order of cofinality } \omega.$$

The extension $L(Q_\omega^{\mathrm{cof}})$ of first order logic by the quantifier Q_ω^{cof} is fully compact ([26]), meaning that it satisfies the Compactness Theorem in vocabularies of any cardinality.

It satisfies also the following strong form of the Downward Löwenheim-Skolem Theorem: Given any model M and a subset X of M of cardinality $\aleph_1$, there is a submodel N of M containing X such that the cardinality of N is $\aleph_1$ and N is an elementary submodel of M in the sense of the logic $L(Q_\omega^{\mathrm{cof}})$. First order logic satisfies the same Downward Löwenheim-Skolem Theorem but with $\aleph_1$ replaced by $\aleph_0$.

By virtue of its model-theoretic properties, $L(Q_\omega^{\mathrm{cof}})$ looks very much like first order logic, but of course it is a proper extension.

Such logics with these nice model-theoretic properties are properly between first and second order logics, however manifesting properties typical of first order logic rather than second order logic. Recent work in inner model theory suggests that these logics really contribute something over and beyond their set-theoretical analogues [15].

Interestingly, some other logics that otherwise are far from first order, behave like first order logic in this inner model context [15].

The Lindström characterization of first order logic mischaracterizes logics in the context of these inner models.

Second complication: Internal categoricity

The prime example of a non-first order logic is second order logic L^2. Can L^2 be seen as a first order logic? As we saw, it is, via symbiosis, a fragment of set theory (i.e. $\Delta_1(\mathrm{Pw})$ in the Levy-hierarchy) and in that sense it can be represented in first order logic if we are granted $\in$.

On the other hand, we can treat L^2 as a two-sorted logic with an individual-sort for elements and a set-sort for sets, relations and functions.[24]

[24] This only makes sense if the Comprehension Schema is assumed in order that L^2 has some second order content. The Comprehension Schema states that every definable (with parameters) set of subsets (or relations or functions) is in the range of the second order variables. There is a natural restriction to prevent circular definitions. For example, without the Comprehension Schema we do not know whether $\forall x \exists X \forall y (yEX \leftrightarrow x = y)$ is valid, although it clearly should be valid.

However, it is not a completely general two-sorted first order logic.

In the two-sorted first order version of L^2 with the Comprehension Schema this is unstable (in the model theoretic sense) and therefore unclassifiable (as a first order theory), again in the sense of stability theory [25].

Just as first order set theory is investigated by the method of transitive models of first order ZFC, second order logic, in its original syntax or alternatively as a two-sorted first order language, can be investigated by the method of Henkin models i.e. sufficiently large collections of sets of urelements, relations between urelements and functions between urelements in order that the Comprehension Schema holds.

What happens to the cherished categoricity results of second order logic, if second order logic is interpreted as first order many-sorted logic? Recent (and in some cases not so recent) results show that categoricity results hold also in the many-sorted version of second order logic, and can be proved from the Comprehension Schemas [34].

They even hold for first order Peano arithmetic and first order ZFC [33]. So the categoricity results of second order logic, despite their smooth formulation in second order logic, turn out to be results about first order logic. The "second-orderness" of *second* order logic is thereby somewhat undermined.

Internal categoricity for first order logic

- Internal categoricity holds also for first order arithmetic and ZFC-set theory, when properly formulated.
- Internal categoricity in first order logic: If two models "know about each other", there is a definable isomorphism between them.
- "know each other" means in arithmetic: the formulas of the Induction Schema of Peano arithmetic can contain non-logical symbols from the other model.

In detail: internal categoricity for first order logic

- A simplification: Suppose $(N, +, \cdot, 0)$ and $(N, +', \cdot', 0')$ satisfy the first order Peano axioms.
- Suppose the Induction Schema of $(N, +, \cdot, 0)$ is stated for formulas in the vocabulary $\{+, \cdot, 0, +', \cdot', 0'\}$ and vice versa.
- Then there is a formula in the vocabulary $\{+, \cdot, 0, +', \cdot', 0'\}$ which defines, provably, an isomorphism between $(N, +, \cdot, 0)$ and $(N, +', \cdot', 0')$.

3. The metatheory problem

If we say that second order logic can express wellfoundedness, we are saying that there is a sentence $\phi(E) \in L^2$ such that for all models (M, E), $E \subseteq M \times M$,

$$(M, E) \text{ is well-founded} \iff (M, E) \models \phi(E). \tag{4}$$

Here the left-hand side is thought as being understood from the "outside", in the metatheory, whatever that means.

In detail: The equivalence (4) informs us, or even defines, the meaning of $\phi(E)$. But what is the meaning of the equivalence (4) *itself*? In particular, what is the meaning of the left hand side of (4)? What criterion are we using to judge whether (M, E) is wellfounded or not on the left side of (4)? If ϕ is the *usual* second order sentence saying that the binary relation E is wellfounded, we can use the same sentence in the left side of (4), except that then (4) becomes a tautology i.e. it says nothing.

Probably most people would say that we should use the (absolute)[25] *set-theoretical* definition of wellfoundedness in the left side of (4).

But how to understand this set theoretical statement on the left hand side of (4)? Barring reference to metatheory, and the

problem of an infinite regress of metatheories, we can understand the meaning of the statement as derived from the axioms of set theory. I.e. we take "(M,E) is well-founded in V" as the criterion

of truth of "(M,E) is well-founded." This is what we do intuitively, but to make the intuition exact we resort to axioms. As to truth in V we say that *at least* what we can derive from the axioms we accept as true in V.

[25] Being well-founded is absolute in set theory.

It is the same with the right hand side of (4). We can derive the meaning of "$(M, E) \models \phi(E)$" from the axioms of second order logic. But then if we use the axioms of second order logic to define the meaning of second order logic, **we are really talking about second order logic as a two-sorted first order logic** or, in other words, second order logic with Henkin semantics, which is completely axiomatizable, rather than second order logic with full semantics.

The horns of our dilemma

So we are forced either to understand the second order statement in terms of first order set theory; or we understand the second order statement as a statement in two sorted first order logic, i.e. second order logic with the Henkin semantics, because we want a logic with a completeness theorem.

Why the second alternative? If we want to use the axiomatic method, then the logic embedding those axioms should be as canonical as possible, i.e. it should have a completeness theorem.

Either way, we fall back into first order logic.

Conclusion

First order logic alone is expressively weak, but when it is combined with $\in$, yielding first order set theory, it is suddenly the strongest logic. Something magical happens when $\in$ is added to the language.

If second order logic is thought of as a "logic" shouldn't we think of first order set theory as a logic, too? If we do, then it is a very high order logic—in fact it is the **strongest** logic.

Via their symbiotic connection we can consider the first order language of set theory to be a many sorted logic—**sort logic**—with a variant of Henkin semantics. The Henkin models of set theory are simply the transitive models that are commonly studied in axiomatic set theory.

Henkin semantics is the *modus operandi* of set theory anyway. . .

As for the question how sharp the FOL/SOL distinction is:

- If we insist on **absoluteness** as a marker of canonicity then many other logics are deemed canonical, along with first order logic.
- If we think of the **Lindström characterization** of first order logic in terms of its model theoretic properties (compactness and Löwensheim-Skolem) as the standard definition of first orderness, then again many other logics have similar model theoretic properties.
- **Internal categoricity** holds of first order Peano and first order ZFC. But isn't categoricity what divides first and second order logic from eachother?
- **The symbiosis story**: First order set theory allows quantification over objects of all orders. But isn't ZFC a first order theory?

Thank you!

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