Jouko 12/14

- $M \models aa\, s\, \phi(s, \bar{a}) \iff \{A \in P_{\omega_1}(M) : (M, A) \models \phi(A, \bar{a})\}$ contains a club.
- Axiomatizable, countably compact
- $Q^{cf}_\omega\, x, y\, \phi(x, y) \iff aa\, s$ ("s is cofinal in $\phi(-,-)$")
- $C(aa)$ is defined using the "J' hierarchy"

$$J' = \emptyset$$

$$J'_{\alpha+\omega} = \mathrm{rud}_{T_r}(J'_\alpha \cup \{J'_\alpha\})$$

$$J'_{\alpha\omega} = \bigcup_{\beta<\alpha} J'_\beta,$$

where $T_r = \{(\beta, \phi(\bar{a})) : (J'_\beta, \in, T_r \upharpoonright \beta) \models \phi(\bar{a}),\ \phi(\bar{a}) \in L(aa),\ \bar{a} \in J'_\beta\}$

$C(aa) = \bigcup_{\alpha \in \mathrm{Ord}} J'_\alpha \models ZFC$

<u>Open Question</u> 1) $C(aa) \overset{?}{=}$ "old" $C(aa) := C_0(aa)$ ← defined w/ Gödel definition allowing $L(aa)$ formulas

2) $C_0(aa) \models AC$?

<u>Lemma</u>

Suppose $\mathbb{P}$ is σ-closed and G is $\mathbb{P}$-generic.

Then $C(aa) = C(aa)^{V[G]}$.

<u>Pf.</u>

Let (J''_α) be the hierarchy for $C(aa)^{V[G]}$.

Then <u>$\forall \alpha\ J''_\alpha = J'_\alpha$.</u>

Requires showing preservation of T_r.

STS: If $S \subseteq P_{\omega_1}(J'_\alpha)$, $S \in V$, then

$(S \text{ stat})^V$ iff $(S \text{ stat})^{V[G]}$.

Club Determinacy (CD):

For any α, any $\phi(s, \bar{x}, t) \in L(aa)$, any $\bar{a} \in J'_\alpha$, any $t \in P_{\omega_1}(J'_\alpha)$:

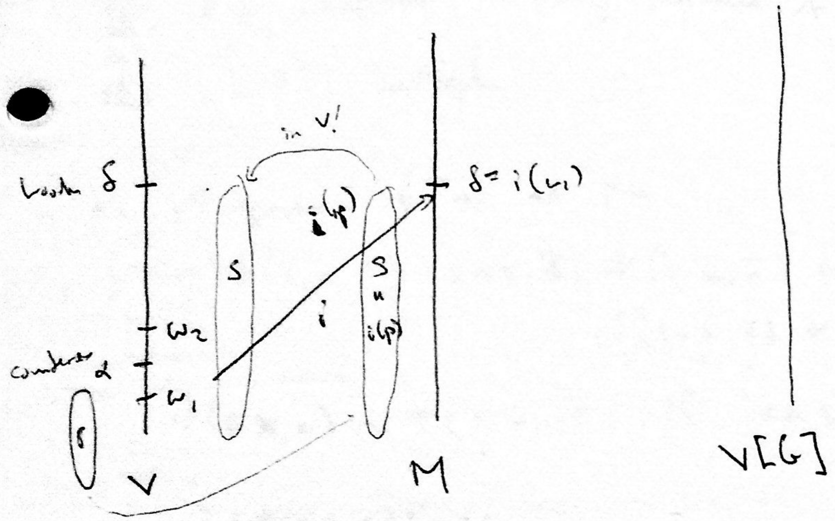
$$J'_\alpha \models aa\, s\, \phi(s, \bar{a}, t) \vee aa\, s\, \neg\phi(s, \bar{a}, t).$$

CD $\Rightarrow$ $\nexists$ stat-co-stat sets.

Theorem

PCWC (or PFA) $\Rightarrow$ CD.

Pf $\quad \delta^1_2 = \omega_2$.



(J'_γ) $\qquad$ (J''_γ)

(J'''_γ) $aa{<}\delta$-hierarchy
$\uparrow$
sets $<\delta$ closed under Skolem functions

Fact

$(J''_\beta)^M = (J''_\beta)^V$, $\beta \in \alpha$

Fact

Sp: $S \subseteq P_{\omega_1}(X)$ stat in $V \Rightarrow S$ stat in $V[G]$.

Corollary Assume PCWC.

1) κ regular uncountable $\Rightarrow$ κ measurable in $C(aa)$

(Club α big. $F = \{X \subseteq \kappa : X \in J'_\alpha,\ J'_\alpha \models aa\, s\, (\sup(s \cap \kappa) \in X)\}$ witnesses.)

2) $Th(C(aa))$ is forcing absolute
($i: V \to M$ stationary tower embedding, $C(aa)^M = C(aa_{<\delta})^V$.)

<u>Theorem</u> Assume CD. Then $C(aa) \models CH$.

<u>Def</u> aa-premouse. $\langle P_i : i<\beta \rangle$ unary predicates

$$(J^T_\alpha, \in, T, T^*, (P)_\beta)$$

1) $T \subseteq \alpha \times L(aa)$, $\beta < \alpha \Rightarrow \{\phi(\bar a) : (\beta, \phi(\bar a)) \in T\}$ is a complete $L(aa)$-theory containing $FO(J^T_\beta, \in, T|\beta)$
2) analogously for T^*, at α.
3) $(P)_\beta$ continuous increasing sequence of unary subsets of J'_α.
4) ... 5) omitted.

aa-ultrapower of an aa-pm

$$M' = \{\phi(s,x,\bar a) : aas\ \exists x\, \phi(s,x,\bar a) \in T^*\}$$
$$\phi(s,x,\bar a) \in FO(aa)$$

$\phi(s,x,a) \sim \psi(s,x,a)$ iff $aas\,(f_{\phi(s,x,a)}(s) = f_{\psi(s,x,a)}(s)) \in T^*$

$M = \{[\phi(s,x,\bar a)] : \phi(s,x,\bar a) \in M'\}$

$[\phi(s,x,\bar a)]\ E\ [\psi(s,x,\bar a)]$ iff $aas\,(f_{\phi(s,x,a)}(s) \in f_{\psi(s,x,a)}(s)) \in T^*$

etc.

$Ult((J'_\alpha, \in, T, T^*, (P)_\beta))$

Łoś' Lemma

level of the model

<u>Lemma</u>
$$\begin{tikzcd}
(J'_\alpha, \in, T, T^*, (P)_\beta) \arrow[rr, "\pi"] \arrow[dr, "i"'] & & (J_\beta, \in, T_\beta, T^*_\beta) \\
& Ult \arrow[ur, dashed, "\pi^*"'] &
\end{tikzcd}$$

<u>Lemma</u> $(J'_\alpha, \in, T, T^*, (P)_\beta)$ and Ult have the same reals.