Jouko 12/12 — Joint w/ J. Kennedy & M. Magidor

$Q^{cf}_\omega xy\, \phi(x,y) \iff \phi(-,-)$ is a LO of cf ω.

Recall

C^* ($= L[\mathrm{cof}\ \omega]$).

If $0^\#$ exists, then $0^\# \in C^*$.

(In fact, $K^{DJ} \subseteq C^*$.)

> **Q** What do we know about $L(\mathbb{R}, Q^{cf}_\omega)$?
> $= C(L_{\omega_1\omega_1})$?
> $= C(L^\omega_{\omega_1\omega_1})$?

Theorem

If there are msbl cardinals, then $V \neq C^*$.

Pf

Let $i : V \to M$, crit $i = \kappa$, ${}^\kappa M \subseteq M$

$$(C^*)^M = C^*,$$

$i : C^* \to C^*$ $\square$

- $\aleph_\omega$ is singular in V of cof ω, but regular in C^*.

Q Is $\aleph_\omega$ regular in C^*? (Assuming $L(s)$).

Theorem (ZFC)

$|P(\omega) \cap C^*| \leq \omega_2^V$.

Pf.

Suppose $a \subseteq \omega$. Construct $(M_\alpha)_{\alpha<\omega_1}$ s.t.

(1) ~~M_α transitive,~~ $|M_\alpha| \leq \omega$, $M_\alpha \prec H_\theta$

(2) $M_\nu = \bigcup_{\alpha<\nu} M_\alpha$, $a \in M_0$

(3) $\beta \in M_\alpha$ and $\mathrm{cof}^V(\beta) = \omega$, then $M_{\alpha+1}$ has a witness

(4) $\beta \in M_\alpha$ and $\mathrm{cof}^V(\beta) > \omega$, then there are unboundedly many

$\gamma > \beta$ ~~and sup $\bigcup M_\alpha \cap \beta$~~ and $\rho \in M_{\gamma+1}$ s.t.

$$\sup\Big(\bigcup_{\xi<\gamma}(M_\xi \cap \beta)\Big) < \rho < \beta$$

$M = \bigcup_{\alpha<\omega_1} M_\alpha$. Let N be collapse of M. [illegible]

Let $\eta = M \cap \mathrm{Ord}$. Then $\eta < \omega_2^V$

$|L_\xi'|^N = L_\xi'$ for $\xi < \eta$. $a \in L_\eta'$. □

<u>Theorem</u>

If there is a Woodin cardinal, then ω_1^V is strongly inaccessible (weakly) in C^*.

<u>Pf.</u>

Sps $\alpha < \omega_1$. Sps $\exists f \in C^*$ s.t. $f : \omega_1^V \to (2^\alpha)^{C^*}$

Take stationary tower embedding $i : V \to M$ w/ ${}^\omega M \subseteq M$

$i(\omega_1) = \delta \leftarrow$ the Woodin

$i(f) : \delta \xrightarrow{1\text{-}1} ((2^\alpha)^{C^*})^M$

$a = i(f)(\omega_1^V) \subseteq \alpha$. $a \notin V$.

OTOH, $(C^*)^M \subseteq (C^*_{<\delta})^V \leftarrow$ cof $<\delta$ quantifier

But $a \in (C^*)^M$, so $a \in V$, $\perp$.

□

<u>Note</u>

$\aleph_{\alpha+1}^V$ is weakly compact in C^* for $\alpha \geq 1$.

<u>Theorem</u>

Sps $\exists$ a Woodin cardinal w/ a msbl above.

Then $\mathbb{R}^{C^*}$ is a ctbl Σ^1_3 set.

<u>Pf.</u>

TFAE: for $a \subseteq \omega$

(1) $a \in C^*$

(2) $\exists$ ctbl transitive iterable model of $ZFC^- + \exists$ Wdn < msbl st $a \in M$, and $M \models$ "$a \in C^*$."

(1) → (2).

By θ, $M \prec H_\theta$ ctbl st a, Woodins, msbl $\in M$

$M \models$ "$a \in C^*$"

$M \cong N$ collapse. N is iterable b/c of the msbl.

(2) → (1).

$$N = M_0 \xrightarrow{\pi_{0,1}} M_1 \xrightarrow{\pi_{1,2}} \cdots \to N_\alpha \to \cdots N_{\omega_1}$$

Note WLOG $a \in L'_\beta$ for some $\beta < \omega_2^V$.

$\pi_{0,\omega_1}(\omega_1^N) = \omega_1^V$.

$\pi_{0,\omega_1}(a) = a$

$(L'_\beta)^{N_{\omega_1}} = L'_\beta$.

Hence $a \in (L'_\beta)^{N_{\omega_1}} = L'_\beta$, so $a \in C^*$.

□

Theorem

Assume PCWC. Then

~~[illegible]~~ If $\mathbb{P}$ is any forcing and G is $\mathbb{P}$-generic, then

(1) $Th(C^*) = Th((C^*)^{V[G]})$

(2) $Th(C^*)$ is independent of the cofinality.

(3) $\mathbb{R}^{C^*} = \mathbb{R}^{(C^*)^{V[G]}}$.

Proof

Let δ be a Woodin w/ $\mathbb{P} \in V_\delta$.

$j_1 : V \to M_1 \subseteq V[H_1]$ stat tower embedding.

$(C^*)^{M_1} = C^*$ b/c j_1 elementary.

Since ${}^\omega M_1 \subseteq M_1$, $(C^*)^{M_1} = (C^*)^{V[H_1]} = (C^*_{<\delta})^V$.

δ is still Woodin in $V[G]$, so $\exists\ j_2 : V[G] \to M_2 \subseteq V[G][H_2]$

$$
\begin{aligned}
(C^*)^{V[G]} = (C^*)^{M_2} = (C^*)^{V[G][H_2]} &= (C^*_{<\delta})^{V[G]} \\
&= C^*_{<\delta}.
\end{aligned}
$$

$\square$