Inner models from extended logics

- $Q_1 x\, \phi(x) \leftrightarrow |\phi(\cdot)| \geq \aleph_1^V$ Introduced by Mostowski

$C(Q_1)$ like L, w/ FOL replaced by $L(Q_1)$,
the extension of FOL w/ $Q_1 x$.

A priori, no reason to expect $C(Q_1) = L$, but

<u>Fact</u> $C(Q_1) = L$.

<u>Pf</u>

$C(Q_1) = \bigcup_{\alpha \in Ord} L'_\alpha$.

By induction $L'_\alpha \in L$ $\forall \alpha$.

Sps $L'_\alpha \in L$, and consider $L'_{\alpha+1}$.

Let $X \in L'_{\alpha+1}$, $X = \{a \in L'_\alpha : L'_\alpha \models \phi(a, \bar{b}^{\in L'_\alpha})\}$.

Ind. on ϕ.

WMA $X = \{a \in L'_\alpha : L'_\alpha \models Q_1 x\, \psi(x, \bar{b}, a)\}$.

Let κ be a cardinal in V s.t. $\kappa > \alpha \geq \aleph_1^V$.

Let $Y_a = \{c \in L'_\alpha : L'_\alpha \models \psi(c, a, \bar{b})\}$

$X = \{a \in L'_\alpha : |Y_a| \geq \aleph_1^V\}$. Then for all $a \in L'_\alpha$:

$L_\kappa \models (\exists \beta < \aleph_1^V\ \exists f : Y_a \xrightarrow{1-1} \beta) \vee (\exists f : \aleph_1^V \xrightarrow{1-1} Y_a)$

$X = \{a \in L_\kappa : L_\kappa \models a \in L'_\alpha \wedge \exists f : \aleph_1^V \xrightarrow{1-1} Y_a\} \in L_{\kappa+1} \subseteq L$.

$\square$

- $W xy\, \phi(x,y) \iff \phi(-,-)$ is a wellorder.

- <u>Fact</u> $C(W) = L$.

<u>Pf</u>.

As before, until

$L_\kappa \models (\exists \beta\, \exists f : W \to \beta$ order preserving$)$

$\quad \vee (\exists f : f$ codes ∞ descending chain in $W)$.

$X = \{a \in L_\kappa : L_\kappa \models a \in L_\alpha' \wedge \exists f \exists \beta\ f : W \xrightarrow{1\text{-}1} \beta\}$

$\in L_{\kappa+1} \subseteq L$.

$\square$

- <u>Fact</u> $C(Q_1, W) = L$.

<u>Cofinality quantifier</u>

$Q^1_\omega xy\, \phi(x,y) \iff \phi(-,-)$ is a linear order of cofinality ω

Introduced by Shelah

Compact

$C^* = C(Q^1_\omega)$

- <u>Fact</u> $C^* \neq L$, if $0^\#$ exists. $\quad (V = L \Rightarrow C(L^*) = L)$

<u>Pf</u> We show:

If $0^\#$ exists, then $0^\# \in C^*$.

Sps I is the canonical club class of indiscernibles for L.

STS there is an infinite $X \subseteq I$ st $X \in C^*$.

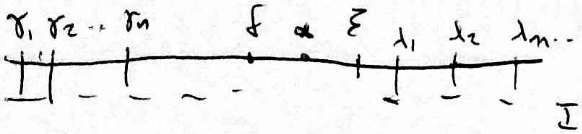
$0^\# = \{\ulcorner \phi(v_1 \ldots v_n) \urcorner : L \models \phi(x_1 \ldots x_n)$ for $x_1 \ldots x_n$ in $I\}$.

$X = \{\alpha < \aleph_\omega : \alpha$ regular in L and $cf^V(\alpha) > \omega\}$.

- <u>Note</u>: $X \in C^*$ b/c we can talk about cof ω.

<u>Claim</u> $X \subseteq I$.

Pf. Let $\xi \in X \setminus I$.



Let $\delta = \max(\xi \cap I)$

Spse $\alpha < \xi$,

$\alpha = \tau(\gamma_1, \ldots, \gamma_n, \lambda_1, \ldots \lambda_m)$

Let $A_\tau = \{\tau(\beta_1, \ldots \beta_n, \lambda_1 \ldots \lambda_m) : \beta_1 \ldots \beta_n < \delta\}$, $A_\tau \in L$.

$|A_\tau| \le \delta < \xi$, $\alpha \in A_\tau$.

Let $\eta = \sup_\tau \sup A_\tau < \xi$, as $\mathrm{cof}\ \xi > \omega$, ξ regular in L.

$\alpha < \xi \Rightarrow \alpha \le \eta$, contradiction.

□

- $C^* = L[\{\alpha \in \mathrm{Ord} : \mathrm{cof}\ \alpha = \omega\}]$.

- ZFC-definable logics construct L.

satisfaction relation abs. for forcing extensions (?) Σ_1 $M \models \phi \leftrightarrow \phi(M, \phi)$

E.g. $L_{\infty,\omega}$, game quantifier, L_{HYP} $\leftrightarrow \psi(M, \phi)$ Π_1

Question $L(\mathbb{R}, \{\alpha : \mathrm{cof}\ \alpha = \omega\}) \models AD$, ϕ formula $\leftrightarrow \theta(\phi)$,

Assuming LCs? If θ Σ_1

($M_1(\mathbb{R})$ contains $\mathbb{R} \cap L(\mathbb{R}, \{\alpha : \mathrm{cof}\ \alpha = \omega\})$.)

- $\mathbb{R} \cap C^* \subseteq M_1$.

Magidor-Malitz quantifier

$$Q_1^{MM} xy\ \phi(x,y) \leftrightarrow \exists X^{\in V}\ |X| \ge \aleph_1 \wedge \forall x,y \in X\ \phi(x,y).$$

- Can express Souslinity, hence highly non-absolute

Fact. 1) $\mathrm{Con}(ZF) \rightarrow \mathrm{Con}(C(Q_1^{MM}) \neq L)$

2) $0^\#$ exists $\rightarrow C(Q_1^{MM}) = L$.

Lemma Spse $C \subseteq \alpha$, $C \in L$, and there is (in V) uncountable B s.t. $[B]^2 \subseteq C$. Then $\exists$ such a B in L.

(Uses Rowbottom property to linearize and obtain B.)