Boban 12/5

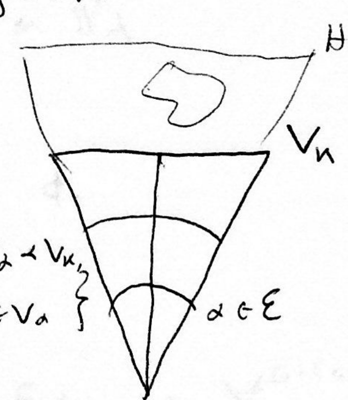
Recall

ϕ infinitary propositional formula in CNF

$\mathbb{H}(\phi)$ p.o. consisting of finite partial assignments

nonempty

Fix inaccessible κ w/ $\phi \in V_\kappa$.



$\mathcal{E} = \{\alpha : V_\alpha \prec V_\kappa, \phi \in V_\alpha\}$

$\lambda \in \mathcal{E} \cup \{\kappa\}$

$\mathbb{P}^*_\lambda$ poset of preconditions

$p = (\mathcal{M}_p, w_p)$

$\mathcal{M}_p = \{M_0 \in M_1 \in \dots \in M_{n-1}\}$

$\lambda_{M_0} < \lambda_{M_1} < \dots < \lambda_{M_{n-1}}$

$C_\alpha = \{M : \text{ctbl}, M \models ZFC, \alpha \in M \prec \hat{M} \cup V_\alpha\}$

$\mathrm{Hull}(M, V_\alpha) \xrightarrow{\pi} \mathrm{Hull}(M, V_\alpha)$

$\pi[M]$

$M \in C_{\lambda_M}$

$w_p \in \mathbb{H}(\phi)$ a finite partial assignment

$q \le p$ iff $w_p \subseteq w_q$ and $\forall M \in \mathcal{M}_p \, \exists N \in \mathcal{M}_q$

$\lambda_M = \lambda_N$, $\underset{M \cap \omega_1}{S_M} = S_N$, $M \prec N$

$\mathbb{P}_\lambda$. $p \in \mathbb{P}^*_\lambda$

Game $G_\lambda(p)$:

I	Q_n
II	

$p_0 = p \ge p_1 \ge p_2 \dots$

I. Q_n: $C \in \phi$

II. $p_{n+1} \le p_n$ ~~s.t.~~ $\exists l \in C$ $l \subseteq p_{n+1}$

OR

I Q_n: (M, D) $M \in \mathcal{M}_{p_n}$, $D \in M$ dense in $\mathbb{P}_{\lambda_M}$

II produces $q \in D$ $\mathrm{Hull}(M, \{q\}) \cap \omega_1 = M \cap \omega_1$ $p_{n+1} \le p_n, q$.

$\mathbb{P}_\lambda = \{p \in \mathbb{P}^*_\lambda : \text{II has w.s. in } G_\lambda(p)\}$

Rmk

$\forall C \in \Phi$ $\{p \in \mathbb{P}_\lambda : \exists \ell \in C, \ell \in \mathrm{ran}\, p\}$ is dense in $\mathbb{P}_\lambda$

(assuming $\mathcal{H}(\Phi) \neq \emptyset$)

—

Forcing w/ $\mathbb{P}_\lambda$ adds a satisfying assignment for Φ.

(Larson) AS-consistency

Given Φ, Φ is AS-consistent if $\forall$ stat $S \subseteq \omega_1$, in $V^{\mathrm{Col}(\omega,\omega_1)}$:

$\forall \mu \models \Phi$ $\exists j : V \to W$, CRT $j = \omega_1^V \in j(S)$

$V^{\mathrm{Col}(\omega,\omega_1)}$ $\exists \hat{\mu} \models_j (\Phi)$ s.t. $j[\mu] \subseteq \hat{\mu}$.

Thm

Φ AS-consistent $\Rightarrow$ $\mathbb{P}_\kappa$ is SSP.

Def

$\mathbb{P} \in M \prec H_\theta$ (ctbl). $\mathbb{P}$ is semiproper for M if $\forall p \in M$ $\exists q \leq p$

q is $(M, \mathbb{P})$-semigeneric

Fact

$\mathbb{P}$ is SSP iff $\{M \prec H_\theta : \mathbb{P}$ is semiproper for $M\}$ is proj. stat.

i.e. for every $S \subseteq \omega_1$ stat, $\{M \prec ; M \cap \omega_1 \in S\}$ is stat.

Def

Suppose $\theta \gg \kappa$ regular, $M \prec H_\theta$ ctbl, $\kappa, \Phi \in M$

M is good if for every $p \in \mathbb{P}_\kappa \cap M$

$\exists q \in \mathbb{P}_\kappa$ $q \leq p$ $\exists \lambda \in E$ $M \restriction \lambda \in M_q$.

$\left(\pi(\kappa) = \lambda, \ \pi : M \to M \restriction \lambda \right.$

$\left. \mathrm{Hull}(M, V_\lambda) \cap V_\kappa = V_\lambda \right)$

Prop

If M is good, then $\mathbb{P}_\kappa$ is semiproper for M.

Rmk

"self-generic" instead of "good"

Sps $r \overset{\mathbb{P}_\kappa}{\le} q$ $\exists N \in M_r$ $M \restriction \lambda \subseteq_{\omega_1} N$, $\delta_N = \delta_{M \restriction \lambda}$

Given $D \in M$ and $s \in D$, $Hull(M, \{s\}) \cap \omega_1 = M \cap \omega_1$

$\exists u \le r, s$

$\pi : M \to M \restriction \lambda$, $\pi(\kappa) = \lambda$, $\pi(D) \in M \restriction \lambda$

I	... $(N, \pi(D))$
II	

$\mathcal{Y} = \{ M \prec H_\theta : M \text{ is good} \}$ is a local club in

$X \prec H_\theta$, $|X| = \omega_1$, $\omega_1 \subseteq X$

Claim $\{M : M \prec X \text{ ctbl, not good}\}$ is not stationary in T

Sps ow.

X is union of M_ξ, $\xi < \omega_1$ $\nearrow$, $M_\xi \cap \omega_1 = \xi$.

Fix $T \subseteq [X]^\omega$ st $p \in X$ and

$\forall M \in T$ $\nexists q \in \mathbb{P}_\kappa$ $q \le p$ ... $M \restriction \lambda \in M_q$ ...
$\nexists \lambda$

Fix $\lambda \in E$ $Hull^{H_\theta}(V_\lambda) \cap V_\kappa = V_\lambda$.

Let h be V-generic / $\mathrm{Col}(\omega_1, \kappa)$.

In $V[h]$, ~~fix~~ g V-generic / $\mathbb{P}_{\dot{x}}$, $p \in g$.

Let $\mu = \mu_g$ be the generic assignment from g.

By AS consistency, $\exists j : V \to W$ (in $V[h]$) w/ crit $j = \omega_1^V$

$\omega_1^V \in j(T)$,

$\exists \hat{\mu} \models j(\phi)$ $\hat{\mu} \supseteq j[\mu]$.

$N = \mathrm{Hull}^{j(H_\theta)}(\omega_1^V \cup \{j(p)\})$

$\prec$

$j(H_\theta)$

$N \in W$, $j(p) \in N$, $N \cap \omega_1^W = \omega_1^V$

$q = (j(\mu_p) \cup \{N \restriction j(\mathcal{L})\}, j(r_p))$