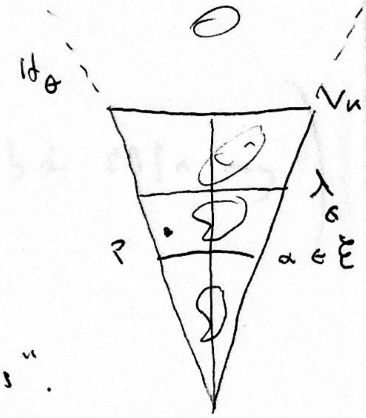
## SSP consistency

Given $\phi \in 2_K$ in CNF

want a satisfying assignment

$\mathbb{H}(\phi)$ the collection of finite partial assignments w/ Player II having winning strategy.

$\mathbb{H}(\phi) = \mathbb{P}_0$




Fix $\kappa$ inaccessible $\gg |\phi|$, and work in $V_\kappa$.

$\mathcal{E} = \{\lambda < \kappa : V_\lambda \prec V_\kappa\}$. club in $\kappa$

$\forall \alpha \in \mathcal{E}$ there will be a collection of "$\alpha$-models".

Conditions will have the form $p = (\mathcal{M}_p, w_p) \in \mathcal{M}$

$M \prec H_\theta$, $\lambda < \kappa$

$Hull(M, V_\lambda) \cap V_\kappa = V_\lambda$

Def $M \restriction \lambda$:

$$\begin{array}{c} Hull(M, V_\lambda) \\ \downarrow \pi \\ \overline{Hull(M, V_\lambda)} \end{array}$$

Then $M \restriction \lambda = \pi[M]$.

Def For $\lambda \in \mathcal{E}$, we define $\mathcal{C}_\lambda$

$M \in \mathcal{C}_\lambda$ iff $M \models ZFC^*$ etc

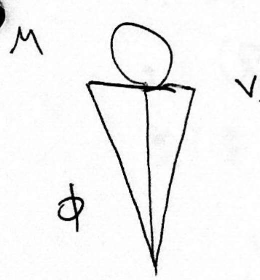
$\phi, V_\lambda, \ldots \in M \prec \hat{M}$ = transitive closure of $M$

$S \subseteq \mathcal{E}$ then $\mathcal{C}_S = \bigcup_{\lambda \in S} \mathcal{C}_\lambda$

$\mathcal{C}_{<\lambda}$

WTS stat subsets of $\omega_1$ are preserved
Given stat set, coloring, and name for club
need to find model and generic condition which
forces that the ~~coloring~~ stat set hits the club.
So the definitions are designed to make the proofs simple

( . $\alpha < \beta$
If $M \in \mathcal{C}_\beta$ then $M \mapsto M \upharpoonright \alpha$ is defined but $M \upharpoonright \alpha$ may not be in $\mathcal{C}_\alpha$ ... )



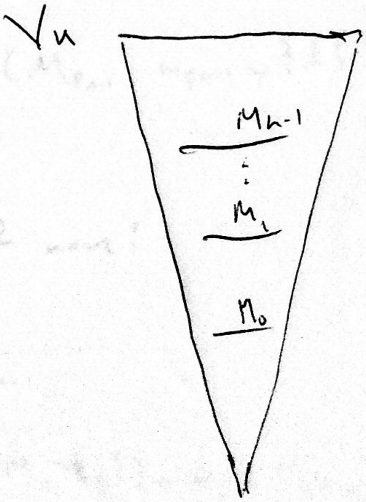

Given a model $M$ there is unique $\lambda_M$ s.t. $M \in \mathcal{C}_{\lambda_M}$

$$\delta_M = M \cap \omega_1$$

Def Preconditions.

$\lambda \in \mathcal{E} \cup \{\kappa\}$.

$\mathbb{P}^*_\lambda$ the candidates for $\mathbb{P}_\lambda$:




$p \in \mathbb{P}^*_\lambda$ if

- $p = (\mathcal{M}_p, w_p)$

$\mathcal{M}_p \subseteq \mathcal{C}_{<\lambda}$, $w_p \in H(\varphi)$

- $M, N \in \mathcal{M}_p$ $\quad \mathcal{M}_p = \{M_0, \ldots, M_{n-1}\}$
  - . $\delta_M = \delta_N \Rightarrow M \cong N$
  - . $\delta_M < \delta_N \Rightarrow \lambda_M < \lambda_N$ & $M \in N$

$p \in \mathbb{P}^*_\lambda$, $\mu < \lambda$

$p | \mu = (\mathcal{M}_p \restriction \mu, w_p)$

$\{M \in \mathcal{M}_p : \lambda_M < \mu\}$

$\mathbb{P}^*_\mu \subseteq \mathbb{P}^*_\lambda$

## To define conditions

Ind assumption

1. $\lambda \in \mathcal{E} \cup \{\kappa\}$
2. $(\mathbb{P}_\xi : \xi \in \lambda \cap \mathcal{E}\}$ is defined
3. $\xi \in \lambda \cap \mathcal{E}$ $\mathbb{P}_\xi \subseteq \mathbb{P}^*_\xi$

$\mathcal{M}_p \subseteq \mathcal{C}_{<\xi}$.

Given $p \in \mathbb{P}^*_\lambda$, define closed game $G_\lambda(p)$

$p_{-1} = p$.

At stage $n$: Have $p_{n-1}$

| I | II |
|---|---|
| $C \in \Phi$ | |
| | $l \in C$ |
| $M \in \mathcal{M}_{p_{n-1}}$ | |
| $D \in M$ | |
| | $q \in D$ |

D dense in $\mathbb{P}_{\lambda_M}$ already defined b/c $\lambda_M <$ current stage

$p_n \leq q, p_{n-1}$

Type 1 move:

$p_n = (\mathcal{M}_{p_{n-1}}, w_{p_{n-1}} \cup \{l\})$

Type 2 move:

$q \parallel p_{n-1}$

$\text{Hull}(M, \{q\}) \cap \omega_1 = M \cap \omega_1 = \delta_M$

$w_{p_n} \in H(\Phi)$

$\mathbb{P}_\kappa = \bigcup_{\lambda \in E} \mathbb{P}_\lambda$

$\mathbb{P}_\lambda$ adds a satisfying assignment of $\phi$.

$\mathbb{P}_\kappa$ is not empty: $p_0 \in \mathbb{P}_\kappa$. ($\psi$ formula in ..., $\psi\downarrow$ is the set of variables in $\psi$)

Q When is $\mathbb{P}_\kappa$ ssp?

**Def** (Kasum)

$\phi$ is AS-consistent if $\forall S \subseteq \omega_1$ stationary

$\forall$ assignments $v : \phi\downarrow \to \{0,1\}$ in $V^{Col(\omega_1, <\kappa)}$, $v \models \phi$

$\exists j : V \to W$, $crit(j) = \omega_1^V \in j(S)$

$\exists \hat{v} : j(\phi)\downarrow \to \{0,1\}$ s.t. $j[v] \subseteq \hat{v}$

$\hat{v} \in V^{Col(\omega_1, <\kappa)}$

j definable here

**Remark**

j could come from $\omega_1$ precipitous, eg.

$$\begin{tikzcd} M \arrow[d, "\pi"] \\ M \restriction \lambda \end{tikzcd} \qquad \pi(\kappa) = \lambda$$

**Def** $M \prec H_\theta$, $\theta > \kappa$, $\phi$ etc $\in M$

M is good if $\qquad Hull(M, V_\lambda) \cap V_\kappa = V_\lambda$

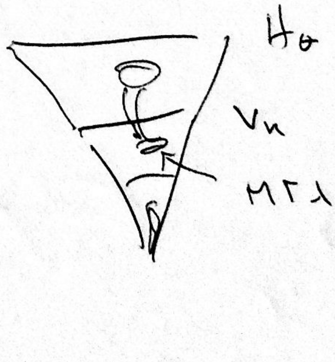
$\forall p \in \mathbb{P}_\kappa \cap M \ \exists \lambda \in E, \lambda < \kappa$ s.t. $\cancel{\kappa \cap M \subseteq \lambda}$

$M \restriction \lambda \in \mathcal{C}_\lambda$

$\exists q \in \mathbb{P}_\kappa \quad M \restriction \lambda \in \mathcal{M}_q$

$\|$

$(\mathcal{M}_p \cup \{M \restriction \lambda\}, w_p) \in \mathbb{P}_\kappa$




**Prop** M is good $\to$ $\mathbb{P}_\kappa$ is semiproper for M.