Boban 11/9

Propositional logic

1930's: Given a first order formula ϕ, decide if $\exists M \models \phi$?

Herbrand: Converted the question $\exists M \models \phi$ to question about satisfiability of propositional theory.

$$\phi \mapsto T_\phi, \text{ where } T_\phi \text{ and } \phi \text{ are equisatisfiable}$$

(We'll do something similar)

$$F \text{ prop. formula} \rightsquigarrow C(F) \quad \text{CNF}$$

Apply resolution to clauses in search of contradiction ie the empty clause.

$$\frac{\begin{array}{c} C_1 \vee p \\ C_2 \vee \neg p \end{array}}{C_1 \vee C_2}$$

What about infinitary versions of this?

Infinitary propositional formulas

Q $\phi \in L_{\kappa\omega}$.

When can we add a model $M \models \phi$ by a forcing in some nice class $\mathcal{K}$?

(Motivated by Asperó - Schindler)

$\phi \in L_\kappa$

propositional variables disjoined / conjoined arb lengths $< \kappa$

Can be converted to CNF, perhaps w/ different prop variables

Looking for satisfying assignment st from each clause, at least 1 variable is true

Forcing axioms can be expressed in this way.

<u>Q</u> Given a set C of clauses, is there $\mathbb{P} \in K$ which adds a satisfying assignment?

If K is the set of all posets, this is to ask for resolution

<u>Games</u>

I	II
$C \ni c_1$	
	$l_1 \in c_1$
$C \ni c_2$	
	$l_2 \in c_2$

II claims to have sat. ass. plays literals from clauses selected by I from C.

I wins iff II plays a literal l_i and its negation

length ω game

Determined.

If I has a winning strategy, it remains winning in any wellfounded model.

If II has a winning strategy and C is ctbl, then C has a satisfying assignment

If C uncountable, collapse $|C|$.

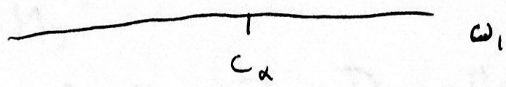
I's strategy remains winning in generic extensions.

<u>Prop</u> II has a winning strategy iff C has a satisfying assignment in some generic extension.

But what if don't want to collapse ω_1?

Ex 1

Can express there is ω-cofinal sequence in ω_1 propositionally.



$$d_n$$

$$\bigvee_\alpha d_n = c_\alpha \qquad \bigvee_n c_\alpha < d_n$$

Cannot be satisfied in ω_1-preserving forcing extension.

Cofinal sequence in ω_2 can be, though (Namba).

Ex 2

Same w/ ω_2 instead of ω_1.

Should reconstruct some form of Namba.

Game starting from p

$$\downarrow p$$

$$G_p(C)$$

I	II
c_1	
	l_1
c_2	
	l_2

C - set of infinitary clauses.

$$\mathbb{H}(C) = \mathbb{P}_0$$

If $p \in \mathbb{H}(C)$ is a finite partial assignment and II has a winning strategy in $G_p(C)$.

$$q \leq p \quad \text{iff} \quad q \supseteq p$$

These posets will collapse $|C|$ to ω...

Recall

$\mathbb{Q}$ is SSP if $\forall E \subseteq \omega_1$ stationary

$$\Vdash_{\mathbb{Q}} \check{E} \text{ is stationary}$$

Will use reformulation

④

$M \prec H_\theta$, θ suff large

q is $(M, \mathbb{Q})$-semigeneric if $q \Vdash \overset{H_\theta[G]}{\vee} M[\dot{G}] \cap \omega_1 = M \cap \omega_1$

or, equivalently,

$\forall r \leq q \ \forall D \in M$ dense $\exists s \in D$ $r \| s$ & $\mathrm{Hull}(M \cap \omega_1, \{s\}) \cap \omega_1 = M \cap \omega_1$

($\mathrm{Hull}(M, \{s\}) = \{f(s) : f \in M$ & $s \in \mathrm{dom}(f)\}$.)

– needs θ has Skolem property

$\mathbb{Q}$ is M-semiproper if $\forall p \in \mathbb{Q} \cap M \ \exists q \leq p$ $(M, \mathbb{Q})$-semigen

$\mathbb{Q}$ is semiproper if for club many M-s ...

Prop $\mathbb{Q}$ is SSP iff $\{M \prec H_\theta : \mathbb{Q}$ is M-semiproper$\} =: \mathcal{Y}(\mathbb{Q})$

is projective stationary in $[H_\theta]^\omega$.

(~~[illegible] $E \subseteq \omega_1$~~ $\searrow$ $\forall F : H_\theta^{<\omega} \to H_\theta$ $\forall E \subseteq \omega_1$ stat

$\exists M \in \mathcal{Y}(\mathbb{Q})$ $\mathrm{cl}_F(M) = M$ & $M \cap \omega_1 = E$.)

— " —

C clauses

$\mathbb{H}(C) = \mathbb{P}_0$

θ suff large

$M \prec H_\theta$ ctbl

Need II to "respect" M

in $G_M(C)$

$G_{M,p}(C)$ $p \in \mathbb{P}_0$

I	II	
$C \ni C_1$	$l_1 \in C_1$	$p_1 = p \cup \{l_1\}$
$A_0 \in M$ max a.c. in $\mathbb{P}_0$	$s_0 \in A_0$ s.t $\mathrm{Hull}(M, \{s_0\}) \cap \omega_1 = M \cap \omega_1$	Add s_0 to p $p_2 = p_1 \cup \{s_0\}$ Remaining in $\mathbb{P}_0$

M is good if II has a.s. for $G_{M,p}(C)$ for all $p \in$ ~~$\mathbb{H}$~~ $\mathbb{P}_0$

Note: clauses I plays need not be in M!

$\mathbb{P}_1$ will consist of $(\underset{\uparrow}{\mathcal{M}}, \underset{\in \mathbb{P}_0}{w})$

Collection of models
consisting of 0, or 1 models

θ will change; need to go to larger θ_1

$H_{\theta_1} > M$