1. <u>Simons's Semester Tutorials</u> – Veličković 9 XI '23

Propositional Logic

<u>1930's</u> – Problem: given a first-order formula ϕ, decide whether there is a model $\mathfrak{A} \models \phi$.

<u>Herbrand</u> "Convert" ϕ to a propositional theory. $\phi \mapsto T_\phi$.

Given a propositional formula F, you can convert it to a formula $C(F)$ in a conjunctive normal form. (CNF).

From CNF, you can try to falsify by <u>resolution</u>.

$$\frac{C_1 \vee p \qquad C_2 \vee \neg p}{C_1 \vee C_2}$$

searching for an empty clause, to show that the original theory is not satisfiable.

Infinitary propositional formulas

$\mathcal{L}_{\infty\omega}$ – you have arbitrarily large $\wedge, \vee$, but can only quantify over finitely variables at a time

<u>Question</u> Given ϕ, $\mathcal{L}_{\kappa\omega}$ sentence when can we add a model $\mathfrak{A} \models \phi$ by a forcing in some class $\mathcal{K}$?

| Motivated by Asperó – Schindler

Given a set of propositional variables P, we can construct a logic $\mathcal{L}_\kappa$ using conjunctions and disjunctions of size $< \kappa$.

We can form infinitary CNF.

2.

<u>Question</u> Given $\mathcal{C}$ - a set of clauses (possibly infinite), is there $\mathbb{P} \in K$ ~~sati~~ adding a ~~set~~, satisfying assignment.

Infinitary resolution still makes sense. (but the process may be transfinite). We can rephrase it in terms of games.

<u>Games</u>

I	II
C_1	
	$l_1 \in C_1$
C_2	
	$l_2 \in C_2$

Player I is Challenger, II is Verifier.
Player I can play a clause which we think of as "which literal is true"
Player II answers with a literal - the one which is supposed to be true.

This game lasts ω-moves. ~~If I wins~~ II wins if there's no contradiction

Proposition

II has a ws iff $\mathcal{C}$ has a satisfying assignment in a generic extension.

<u>Proof</u>

Collapse the size of $\mathcal{C}$, to ω and ask about all the clauses. A strategy for II remains a strategy

<u>Example 1</u>

You can say: "ω is cofinal in ω_1" with a propositional formula, so It will have a satisfying assignment ~~iff it's~~ only in forcing extensions collapsing ω_1.

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Example 2

Same with ω_2 instead of ω_1. You can ~~do~~ cod[?] a satisfying set by SSP forcing (stationary set preserving).

Let $\mathcal{C}$ – set of infinitary clauses.

Take ~~H~~ $H(\mathcal{C})$, a forcing s.t. $p \in H(\mathcal{C})$ if p is a finite partial assignment, and II has a winning strategy in $G_p(\mathcal{C})$, the game $G(\mathcal{C})$ starting with a fixed assignment p.

$$q \leq p \quad \text{iff} \quad q \supseteq p.$$

Recall A poset $\mathbb{Q}$ is SSP iff for every $E \subseteq \omega_1$ stationary $\Vdash_{\mathbb{Q}} \check{E}$ is stationary.

If $M \prec H_\theta$, ~~q~~ a condition q is $(M, \mathbb{Q})$-semigeneric if $q \Vdash M[\dot{G}] \cap \omega_1 = M \cap \omega_1$.

Equivalently for every $r \leq q$, for every $D \in M$ dense, there's a condition s in D, ~~$\Vdash r$~~ $r \parallel s$, s.t.

$$\mathrm{Hull}(M, \{s\}) \cap \omega_1 = M \cap \omega_1.$$

Note that ~~to take a hull it's enough to~~

$$\mathrm{Hull}(M, \{s\}) = \{ f(s) : f \in M, \ \cancel{\mathrm{dom}\, s \in} \ s \in \mathrm{dom} f \}.$$

Now, a poset $\mathbb{Q}$ is M-semiproper (for a single M) if $\forall p \in \mathbb{Q} \ \exists q \leq p$ $(M, \mathbb{Q})$-semigeneric

4. Proposition

$\mathbb{Q}$ is SSP iff $\{M \prec H_\theta : \mathbb{Q}$ is M-semiproper$\}$ is projective stationary in $[H_\theta]^\omega$.

| Stationary and given $E \subseteq \omega_1$ stationary, we can find M s.t. $M \cap \omega_1 \in E$

We'll consider a new kind of games: for $M \prec H_\theta$ cblt. and $p \in \mathbb{P}_0$

$$G_{M,p}(\mathcal{C})$$

I	II
$C_1 \in \mathcal{C}$	
	$l_1 \in C_1$
$\mathcal{A}_0$	
	$s_0 \in \mathcal{A}_0$

I can play either clauses or $\mathcal{A}$-maximal antichains in $\mathbb{P}_0$ from M.

~~lit ex~~ II extends ~~both~~ p both with literals and s_i. | This has to remain consistent

Moreover, we require that $Hull(M, \{s_n\}) \cap \omega_1 = M \cap \omega_1$.

M is a good model if II has a w.s. for any $p \in M$, $p \in \mathbb{P}_0$.

| Note: clauses do not have to come from M

$\mathbb{P}_1$ consists of conditions ~~(M, w)~~ $(\mathcal{M}, w)$, where $\mathcal{M}$ is a collection of models which has at most 1 model.