1. Simons Tutorial – Steel 23 X 23

Based on 1. Mouse pairs and Suslin cardinals (2020?)

2. Mouse ~~pairs~~ limits on Suslin cardinals (Jackson–Sargsyan–Steel)

3. Mouse pairs and Suslin cardinals in type 1 hierarchy (handwritten notes on request)

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(1) Mouse pairs / usually our background theory: AD^+.

They are pairs of the form (P, Σ), where • P is a ctbl transitive premouse (strategy or pure extender premouse), and Σ is an (ω_1, ω_1)-iteration strategy for P, nice in the sense:

(1) Σ has strong hull condensation:

roughly: if you take a Skolem hull $\Phi : \mathcal{S} \to \mathcal{T}$ in a tree (with careful definition what it is) and $\mathcal{T}$ is acc. to Σ, then $\mathcal{S}$ is acc. to Σ.

(2) Σ normalizes well.

For stacks $\mathcal{T} \lhd \mathcal{U}$, there exists a normal tree $W(\mathcal{T}, \mathcal{U})$ and the last model of $\mathcal{U}$ embeds into the last model of $W(\mathcal{T}, \mathcal{U})$

• if $\mathcal{T}, \mathcal{U}$ are by Σ, then $W(\mathcal{T}, \mathcal{U})$ by Σ.

(3) Internal lift consistency

(4) If P is ~~the~~ a strategy premouse, (P, Σ) is pushforward consistent: If ~~$P \xrightarrow{\mathcal{T}} N$ is by Σ~~

2. If $P \overset{\vec{\mathcal{T}}}{<<<\to} N$ is by Σ

then the tail strategy $\Sigma_{\vec{\mathcal{T}}, N}$ is given by:

$$\Sigma_{\vec{\mathcal{T}}, N}(t) = \Sigma(\vec{\mathcal{T}} ^\frown (t))$$

Pushforward consistency is:

if $\vec{\mathcal{T}}$ is by Σ with the last model N, then

$$\dot{\Sigma}^N \subseteq \Sigma_{\vec{\mathcal{T}}, N}$$

(where $\dot{\Sigma}^N$ - strategy given by ~~by~~ the strategy provided by N)

We can compare mouse pairs of the same type

(i.e. pure extender strategy):

$$\begin{tikzcd}
(P, \Sigma) \arrow[dr, "\mathcal{T}"] & \\
& (R, \Psi) \\
(Q, \Lambda) \arrow[ur, "\mathcal{U}"'] &
\end{tikzcd}$$

where $\mathcal{T}$ by Σ, $\mathcal{U}$ by Λ, and Ψ $\Sigma_{\mathcal{T}, R} = \Psi, \Lambda_{\mathcal{U}, R}$.

and at most one of the branches drops.

We also have Dodd-Jensen

$(P, \Sigma) \leq^* (Q, \Lambda)$ iff P to R does ~~d~~ not drop.

$\leq^*$ is called mouse-order. ~~H~~

DJ: $\cdot$ $\leq^*$ is a prewellorder

$\cdot$ Also: if

$$\begin{tikzcd}
(P, \Sigma) \arrow[r, "\text{stack by } \Sigma", "i_s"'] & (R, \Lambda) \\
& (P, \Sigma) \arrow[u, "k"]
\end{tikzcd}$$

only initial segment ?

$k : P \longrightarrow R$

and

$\Lambda^k = \Sigma$

k pullback of Λ

$\Lambda^k(\mathcal{T}) = \Lambda(k\mathcal{T})$

then: $i_s(\eta) \leq k(\eta)$ for all η

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Corollary $\exists$ at most one iteration map from (P,Σ) to (R,Λ).

Because of this, we can define mouse limits. ~~Let~~ For (P,Σ) a mouse pair, let $\mathcal{F}(P,\Sigma) = \{(Q,\Lambda) : (Q,\Lambda)$ is a non-dropping iterate of (P,Σ) by ~~a~~ countable stack$\}$

By comparison DD:

$$\begin{tikzcd}
 & (Q_1,\Lambda_1) \arrow[dr] & \\
(P,\Sigma) \arrow[ur] \arrow[dr] & & (R,\Psi) \\
 & (Q_2,\Lambda_2) \arrow[ur] &
\end{tikzcd}$$

$\mathcal{F}(P,\Sigma)$ is a directed system, so it has a direct limit. Let

$$M_\infty(P,\Sigma) = \text{the direct limit of } \mathcal{F}(P,\Sigma).$$

~~We have~~

Proposition ~~M_∞~~ $(P,\Sigma) \overset{\text{mouse equivalent}}{\equiv^*} (Q,\Lambda)$ iff $M_\infty(P,\Sigma) = M_\infty(Q,\Lambda)$.

Corollary Each $M_\infty(P,\Sigma)$ is ordinal definable (from the place in the mouse ordering). In fact it is in HOD.

Moreover, $\alpha \mapsto H_\alpha$ — common $M_\infty(P,\Sigma)$ for (P,Σ) of mouse rank α. is in HOD.

So $AD_{\mathbb{R}}$ is in $HOD = L[\alpha \mapsto H_\alpha]$.

Suslin Cardinals

Definition A is κ-Suslin ($A \subseteq \mathbb{R} = \omega^\omega$) iff $A = p[T]$ for some T on $\omega \times \kappa$. | $x \in p[T]$ iff for some $y \in \kappa^\omega$ $(x\restriction n, y \restriction n) \in T$ for all n.

κ is a Suslin Cardinal iff $\exists A \subseteq \mathbb{R}$ s.t. A is κ-Suslin but not λ-Suslin for $\lambda < \kappa$.

4. $S(\kappa) = \{A \subseteq \mathbb{R} : A \text{ is } \kappa\text{-Suslin}\}$.

$S(\kappa)$, is closed downwards under $\le_w$, closed under $\exists^{\mathbb{R}}$

, is closed under countable intersections.

| not necessarily closed under complements or $\forall^{\mathbb{R}}$.

↓

closed under ¬ iff κ is not Suslin

Thm (Woodin) Under AD^+ Suslin cardinals are closed set in Θ.

Suslin cardinals + mouse limits in the projective hierarchy.

~~$\kappa_5 = \underset{\sim}{\delta}^1_5 = \kappa_4^+$~~

$\kappa_4 =$ ~~Env_3~~ δ_{Env_3}

$\kappa_3 = \kappa_2^+ = \underset{\sim}{\delta}^1_3$

$\kappa_2 =$ ~~Env~~ $\underset{\sim}{\delta}_{Env_2}$

$\kappa_2 = \aleph_\omega$

$\kappa_1 = \underset{\sim}{\delta}^1_1 = \omega_1$

$\kappa_0 = \omega$ κ_i-Suslin card.

Scale and pwo properties in circles:

$\underset{\sim}{\Pi}^1_4$ $\underset{\sim}{\Sigma}^1_4$

$\underset{\sim}{\Pi}^1_3$ $\underset{\sim}{\Sigma}^1_3$

$\underset{\sim}{\Pi}^1_2$ $\underset{\sim}{\Sigma}^1_2$ $- S(\kappa_1)$

$\underset{\sim}{\Pi}^1_1$ $\underset{\sim}{\Sigma}^1_1$ $- S(\kappa_0)$

$\underset{\sim}{\Pi}^0_1$ $\underset{\sim}{\Sigma}^0_1$

envelope

$Env(\Sigma^1_1 \cup \Pi^1_1) =$

| Envelopes - apparently a standard definition

$Env_2 = Env(\underset{\sim}{\Sigma}^1_2 \cup \Pi^1_2)$, $\underset{\sim}{\delta}_{Env_2}$ - prewellordering ordinal for Env_2.

Fact ~~$\kappa_2 = ord(M_\infty$~~ ~~$\kappa_2 = ord(M_1 | \delta_1^{M_1}, \Sigma_{M_1})$~~

P_1

$\kappa_2 = ord(M_\infty(M_1 | \delta_1^{M_1}), \Sigma_{M_1})$

if $\beta_\infty = \pi_{P,\infty}$ (least strong to δ_1 in M_1), then $|\beta_\infty| = \omega_1$

$\kappa_4 = ord(M_\infty(M_3 | \delta_1^{M_3}, \Sigma_{M_3})$

$\kappa_3 = |\beta_\infty|$, where β_∞ - π least strong to κ_4 in M_∞