

Based on d. Mouse pairs and Suslin cardinals (2020?)

2. Mouse pairs limits on Suslin cardinals (Jackson-Sangya-Steel)

3. Mouse pairs and Suslin cardinals in type 1 hierarchy (handwritten notes on request)

(1) Mouse pairs | usually our background theory: AD^+ .

They are pairs of the form (P, Σ) , where P is a \forall hull transitive premouse (strategy or pure extender premouse), and Σ is an (ω_1, ω_1) -iteration strategy for P , nice in the sense:

(1) Σ has strong hull condensation:

roughly: if you take a Skolem hull $\bar{D}: \mathcal{J} \rightarrow \mathcal{I}$ in a tree (with careful definition what it is) and \mathcal{I} is acc. to Σ , then \mathcal{J} is acc. to Σ .

(2) Σ normalizes well.

For stacks $\mathcal{J} \triangleleft \mathcal{U}$, there exists a normal tree

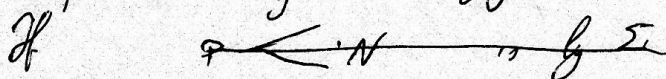
$W(\mathcal{J}, \mathcal{U})$ and the last model of \mathcal{U} embeds into the last model of $W(\mathcal{J}, \mathcal{U})$

• if \mathcal{J}, \mathcal{U} are by Σ , then $W(\mathcal{J}, \mathcal{U})$ is by Σ .

(3) Internal lift consistency

(4) If P is a strategy premouse, (P, Σ) is pushforward

consistent:



2. If $P \ll \ll N$ is by Σ
 then the tail strategy $\Sigma_{\bar{J}, N}$ is given by:

$$\Sigma_{\bar{J}, N}(t) = \Sigma(\bar{J}(t))$$

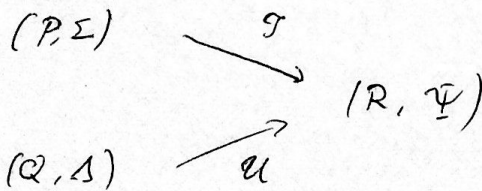
Pushforward consistency is:

if \bar{J} is by Σ with the lost model N , then

$$\Sigma^N \subseteq \Sigma_{\bar{J}, N}$$

where Σ^N -strategy given by the strategy provided by N

We can compare mouse pairs of the same type
 (i.e. pure extender strategy):



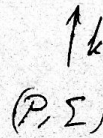
where \mathcal{J} by Σ , \mathcal{U} by Λ , and $\Psi \Sigma_{\mathcal{J}, R} = \mathcal{U}, \Lambda \mathcal{U}, R$.
 and at most one of the branches drops.

We also have Dodd-Jensen

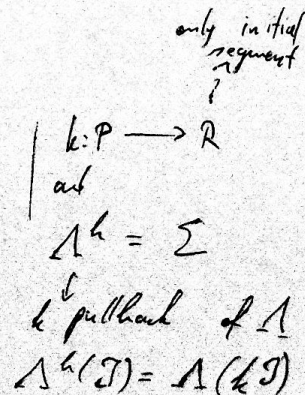
$(P, \Sigma) \leq^* (Q, \Lambda)$ iff P to R does not drop.
 \leq^* is called mouse-order. \mathcal{J}

DJ: $\cdot \leq^*$ is a prewellorder

Also: if $(P, \Sigma) \xrightarrow[\text{is}]{\text{stack by } \Sigma} (R, \Lambda)$



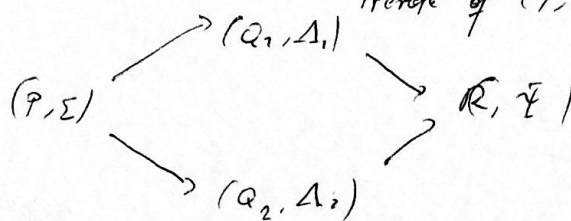
then: $\text{is}(y) \leq k(y)$ for all y



Condition \exists at most one iteration map. from (P, Σ) to (R, Λ) .

Because of this, we can define mouse limits. Let $\text{For}(P, \Sigma)$ a mouse pair, let $\mathcal{F}(P, \Sigma) = \{ (Q, \Lambda) : (Q, \Lambda) \text{ is a non-stopping iterate of } (P, \Sigma) \text{ by a countable stack} \}$

By DD :
comparison



$\mathcal{F}(P, \Sigma)$ is a directed system, so it has a direct limit. Let

$M_\alpha(P, \Sigma) =$ the direct limit of $\mathcal{F}(P, \Sigma)$.

We have

Proposition $M_\alpha(P, \Sigma) \stackrel{?}{\equiv}^* (Q, \Lambda)$ mouse equivalent iff $M_\alpha(P, \Sigma) = M_\alpha(Q, \Lambda)$.

Corollary Each $M_\alpha(P, \Sigma)$ is ordinal definable (from the place in the mouse ordering). In fact it is in HOD .

Moreover, $\alpha \mapsto \text{H}_{\alpha}$ - canonical $M_\alpha(P, \Sigma)$ for (P, Σ) of mouse rank α . is in HOD .

So $\text{AD}_{\mathbb{R}}$ is in $\text{HOD} = L[\alpha \mapsto \text{H}_{\alpha}]$.

Suslin Cardinals

Definition A is \aleph_1 -Suslin ($A \subseteq \mathbb{R} = \omega^\omega$) iff $A = p[T]$ for some T on ω^{\aleph_1} .
 $\aleph_1 \in p[T]$ iff for some $y \in \omega^{\aleph_1}$
 $(x_n, y_n) \in T$ for all n .

\aleph_1 is a Suslin Cardinal iff $\exists A \subseteq \mathbb{R}$ s.t. A is \aleph_1 -Suslin but not \aleph_2 -Suslin for $\aleph_2 < \aleph_1$.

4. $S(\aleph) = \{A \in \mathbb{R} : A \text{ is } \aleph\text{-Suslin}\}$.

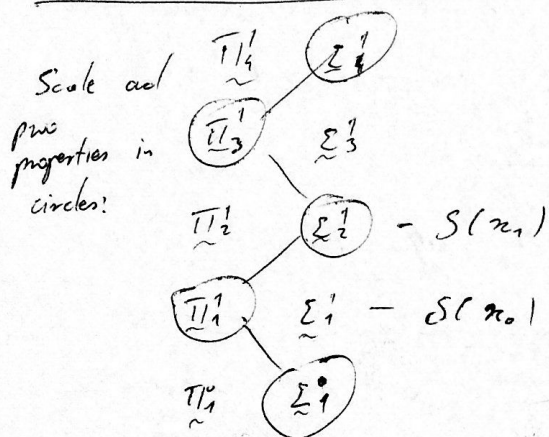
$S(\aleph)$ is closed downwards under \leq , closed under $\exists^{\mathbb{R}}$
 is closed under countable intersections.

Not necessarily closed under complements or $\forall^{\mathbb{R}}$.

closed under \neg iff \aleph is not Suslin

Thm (Washin) Under AD^+ Suslin cardinals are closed set in \aleph .

Suslin cardinals + mouse limits in the projective hierarchy.



$$\begin{aligned} \aleph_4 &= \aleph_3^+ = \aleph_4^+ \\ \aleph_4 &= \text{Env}_3 \delta \text{Env}_3 \\ \aleph_3 &= \aleph_2^+ = \aleph_3^+ \\ \aleph_2 &= \text{Env}_2 \delta \text{Env}_2 \\ \aleph_2 &= \aleph_\omega \\ \aleph_1 &= \aleph_1^+ = \omega_1 \\ \aleph_0 &= \omega \end{aligned}$$

\aleph_i - Suslin card.

\nearrow envelope

$$\text{Env}(\Sigma_1^1 \cup \Pi_1^1) =$$

| Envelopes - apparently a standard definition

$$\text{Env}_2 = \text{Env}(\Sigma_2^1 \cup \Pi_2^1) \quad \delta_{\text{Env}_2} - \text{prewellordering ordinal for Env}_2$$

~~Part $\aleph_2 = \text{ord}(M_{\aleph_0} \quad \aleph_2 = \text{ord}(M_{\aleph_1} / \delta_{\aleph_1}^{M_{\aleph_1}}, \Sigma_{M_{\aleph_1}})$~~

$$\aleph_2 = \text{ord}(M_{\aleph_0}(M_{\aleph_1} / \delta_{\aleph_1}^{M_{\aleph_1}}), \Sigma_{M_{\aleph_1}})$$

if $\beta_{\aleph_0} = \Pi_{\beta, \aleph_0}$ (least strong to δ_{\aleph_1} in M_{\aleph_1}), then $|\beta_{\aleph_0}| = \omega_1$

$$\aleph_4 = \text{ord}(M_{\aleph_0}(M_{\aleph_3} / \delta_{\aleph_3}^{M_{\aleph_3}}), \Sigma_{M_{\aleph_3}})$$

$$\aleph_3 = |\beta_{\aleph_0}| \text{ where } \beta_{\aleph_0} = \pi \text{ least strong to } \aleph_4 \text{ in } M_{\aleph_1}$$