Successor Cardinals in Determinacy Models

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Under AD successor cardinals need not be regular. On the other hand they may posess “large cardinal” like properties such as measurability, supercompactness, and various infinite exponent partition properties.

Within the projective ordinals, the measure analysis gives a detailed understanding of the successor ordinals including their cofinalities and the partition properties of the regular cardinals.

The cardinal structure below the projective ordinals can be described either though the measure analysis using “descriptions” or with a purely algebraic manner using “ordinal algebras”.

We recall some of these results within the projective ordinals.

- The δ^1_{2n+1} are all regular cardinals and have the strong partition property.
- There are $2^{n+1}-1$ regular cardinals strictly between δ^1_{2n+1} and δ^1_{2n+3}, one of which is $\delta^1_{2n+2}=(\delta^1_{2n+1})^+$.
- $\delta^1_{2n+1}=\aleph_{e_n+1}$, where $e_0=0$ and $e_{n+1}=\omega^{\omega^{e_n}}$.
- $\sup_n \delta^1_{2n+1}=\aleph_{\epsilon(0)}$.

The regular cardinals up to δ^1_7 are:

$\aleph_0$	$\delta^1_1 = \aleph_1$	$\delta^1_2 = \aleph_2$	$\delta^1_3 = \aleph_{\omega+1}$
$\delta^1_4 = \aleph_{\omega+2}$	$\aleph_{\omega\cdot 2+1}$	$\aleph_{\omega^\omega+1}$	$\delta^1_5 = \aleph_{\omega^{\omega^\omega}+1}$
$\delta^1_6 = \aleph_{\omega^{\omega^\omega}+2}$	$\aleph_{\omega^{\omega^\omega}+\omega+1}$	$\aleph_{\omega^{\omega^\omega}+\omega^\omega+1}$	$\aleph_{\omega^{\omega^\omega}\cdot 2+1}$
$\aleph_{\omega^{\omega^{\omega+1}}+1}$	$\aleph_{\omega^{\omega^{\omega\cdot 2}}+1}$	$\aleph_{\omega^{\omega^{\omega^\omega}}+1}$	$\delta^1_7 = \aleph_{\omega^{\omega^{\omega^{\omega^\omega}}}+1}$

Figure: The regular cardinals up through δ^1_7.

Either the description analysis or the ordinal algebra analysis gives a formula/algorithm for computing the cofinalities of the successor cardinals below the projective ordinals.

For example, below δ^1_5 we have the following computation of the cofinalities of the successor cardinals.

Theorem

Suppose $\delta^1_3 = \aleph_{\omega+1} < \aleph_{\alpha+1} < \aleph_{\omega^{\omega^\omega}+1} = \delta^1_5$. *Let* $\alpha = \omega^{\beta_1} + \cdots + \omega^{\beta_n}$, *where* $\omega^\omega > \beta_1 \geq \cdots \geq \beta_n$ *be the normal form for* α. *Then:*

- *If* $\beta_n = 0$, *then* $\text{cof}(\kappa) = \delta^1_4 = \aleph_{\omega+2}$.
- *If* $\beta_n > 0$, *and is a successor ordinal, then* $\text{cof}(\kappa) = \aleph_{\omega\cdot 2+1}$.
- *If* $\beta_n > 0$ *and is a limit ordinal, then* $\text{cof}(\kappa) = \aleph_{\omega^\omega+1}$.

As another example (where $\delta^1_5 < \kappa < \delta^1_7$):

Let $\kappa = \aleph_{\alpha+1}$ where $\alpha = \omega^{\omega^{(\omega^{\omega^2}+\omega^{\omega\cdot 2+3})}}$.

Then

$$\mathrm{cof}\left(\aleph_{\omega^{\omega^{(\omega^{\omega^2}+\omega^{\omega\cdot 2+3})}}+1}\right) = \aleph_{\omega^{\omega^{\omega\cdot 2}}+1}.$$

These cofinality computations agree with an earlier result of Kechris and Woodin using the theory of generic codes:

Theorem (Kechris-Woodin)

For all successor cardinals λ^+ below the supremum of the projective ordinals, cof(λ^+) $> \kappa$ where κ is the largest suslin cardinal below λ^+.

Conjecture

Assume AD$^+$. Then for every successor cardinal $\lambda^+ < \Theta$, cof$(\lambda^+) > \kappa$ where κ is the largest Suslin cardinal $< \lambda^+$.

There are other conjectures about successor cardinals in determinacy models.

Conjecture

Assume AD^+. Then there does not exists a cardinal $\kappa < \Theta$ such that $\kappa, \kappa^+, \kappa^{++}$ and κ^{+++} are all regular. More generally, there cannot exist three consecutive regular successor cardinals.

Aside from being central to the measure analysis, partition relations are important for understanding the behavior of determinacy models.

Theorem (Kechris-Woodin)

$L(\mathbb{R})$ satisfies AD iff there are arbitrarily large $\kappa < \Theta$ with $\kappa \to \kappa^\kappa$.

We have the following recent result.

Theorem (Chan, J, Trang)

Assume $\kappa \to \kappa^\kappa$. Then every $\Phi \colon \kappa^\kappa \to On$ is monotonically increasing almost everywhere.

A similar result holds for the weak partition property.

A related result is the following.

Theorem (Chan, J, Trang)

Assume $\kappa \to \kappa^{<\kappa}$. Let $\epsilon < \kappa$ with cof$(\epsilon) = \omega$. Then if $\Phi \colon \kappa^\epsilon \to On$, there there is a $\delta < \epsilon$ such that almost everywhere $\Phi(f)$ depends only on $f \restriction \delta$ and $\sup(f)$.

A corollary of this result is the following.

Theorem

Assume $\kappa \to \kappa^{<\kappa}$. Then there not exist an injection from $\kappa^{<\kappa} \to On^\lambda$ for any $\lambda < \kappa$.

Below the projective ordinals, there are no successor of singular cardinals which have the weak but the strong partition property.

- In fact, the only cardinals having the weak but not the strong partition property are the δ^1_{2n+2}.

Assuming AD, there are arbitrarily large $\kappa < \Theta$ with the strong partiton property [Kechris, Kleinberg, Moschovakis, Woodin], but these are not successor cardinals.

Question

Assume AD^+. Are there arbitrarily large successor cardinals below Θ that have the strong partition property?

Question

Assume AD^+. Are there arbitrarily large successor cardinals λ^+ below Θ that have the weak partition property? Can we control $\mathrm{cof}(\lambda)$?

Positive partiton results

Theorem (Chan, J, Trang)

Let κ be a regular limit Suslin cardinals, and let μ be a measure on κ. Then:

1. *$j_\mu(\kappa)$ is the supremum of successor cardinals λ^+ where cof$(\lambda) = \omega$ and λ^+ has the $< \omega_1$ partition property.*
2. *If μ is a "reasonable" normal measure, then $j_\mu(\kappa)$ is the supremum of successor cardinals λ^+ where cof$(\lambda) = \omega$ and $\lambda^+ \to (\lambda^+)^{<\kappa^+}$.*

In particular, $j_\mu(\kappa)$ is the limit of regular successor cardinals.

Corollary

Assume $\mathsf{AD} + V = L(\mathbb{R})$. Then Θ is the limit of successor cardinals λ^+ for which $\lambda^+ \to (\lambda^+)^{\delta^2_1}$.

The above results are obtained by established a polarized form of the partition property. Arguments borrow from Apter, J, Löwe.

Theorem (CJT)

Let κ be a regular limit Suslin cardinal. Let μ be a measure on κ and let $g\colon \kappa \to \kappa$ with $g(\alpha)$ a limit Suslin cardinal with $g(\alpha)$ of uniform cofinality ω (with $[g]_\mu$ sufficiently large). Let $h\colon \kappa \to \kappa$ with $h(\alpha) \in (g(\alpha), g(\alpha)^+)$, so $[h]_\mu < \lambda^+$, where $\lambda = [g]_\mu$. Let $\mathcal{P}$ be a partition of the $h(\alpha)$ block functions (order-type $h(\alpha)$ in $(g(\alpha), g(\alpha)^+)$ and of the correct type). Then there is a block c.u.b. sequence $\{C_\alpha\}_{\alpha<\kappa}$ which is homogeneous for $\mathcal{P}$.

The following result relates the polarized partition relations with the partition properties at λ^+.

Theorem (CJT)

Let κ, $g\colon \kappa \to \kappa$, and $\lambda = [g]_\mu$ be as above (μ an arbitrary measure). Then for any $\rho < \lambda^+$ we have that $\lambda^+ \to (\lambda^+)^\rho$ iff $(\lambda^+)^\rho \subseteq Ult_\mu$.

Theorem

With κ, μ, g, λ as above, if μ is a "reasonable" normal measure and $\rho < \kappa^+$, then $(\lambda^+)^\rho \subseteq Ult_\mu$.

Negative partition results

Theorem (CJT)

Let κ be a regular limit Suslin cardinal, μ the ρ-cofinal normal measure on κ, Let $\lambda_0 = [g_0]_\mu < [g_1]_\mu = \lambda_1 < j_\mu(\kappa)$ with g_0, g_1 as above (uniform cofinality ω). Then $\lambda_1^+ \not\rightarrow (\lambda_1^+)^{\lambda_0^+}$.

Theorem (CJT)

Let κ be a regular limit Suslin cardinal, μ the ω-cofinal normal measure on κ. Let $\lambda = [g]_\mu$ as before (g of uniform cofinality ω). Then $\lambda^+ \not\rightarrow (\lambda^+)^{\kappa^+}$.

Question

Assume AD^+. Is Θ the limit of successor cardinals which have the strong partition property?

Question

Assume AD^+. Let κ be a regular limit Suslin cardinals. Let $\rho < j_\mu(\kappa)$ be regular. Is $j_\mu(\kappa)$ the limit of successor cardinals λ^+ with $\text{cof}(\lambda^+) = \rho$?

Question

Assume $AD + V = L(\mathbb{R})$. For every successor cardinal $\lambda^+ > \boldsymbol{\delta}^2_1$, do we have $\text{cof}(\lambda^+) > \boldsymbol{\delta}^2_1$?

Return to the conjecture that for $\kappa < \lambda^+$, κ a Suslin cardinal, we have $\mathrm{cof}(\lambda^+) > \kappa$.

In considering this question, another notion arises.

Definition

Let μ be a measure on a set X. We say μ has the δ covering property if for any $f\colon X \to \alpha < \Theta$, there is a set $A \subseteq X$ with $\mu(A) = 1$ and $|f \restriction A| \leq \delta$.

We say the measure μ on X has essential cardinality δ if it has the δ covering property but not the δ' covering property for any $\delta' < \delta$.

We have the following results about essential cardinality.

Theorem (Chan, J, Trang)

Let μ be the supercompactness measure on $\mathcal{P}_{\omega_1}(\kappa)$ for κ a Suslin cardinal. Then μ has essential cardinality $\leq \kappa$.

Theorem (Chan, J, Trang)

Let μ be the strong partition measure on the function space κ^κ. Then μ has essential cardinality $\leq \kappa$.

On the other hand we have:

Fact

The Martin measure on the Turing degrees $\mathcal{D}$ does not have essential cardinality $< \Theta$.

The connection between essential cardinalities and the cofinality conjecture is the following.

The next result says that is we can get a “uniform” version of the covering result for $\mathcal{P}_{\omega_1}(\kappa)$, then the cofinality conjecture holds.

Definition

We say the measure μ on X has λ-uniform δ-covering if for every $F \colon X \times \lambda \to \mathrm{On}$ there is a sequence $\{A_\beta\}_{\beta<\lambda}$ of μ measure one subsets of X such that for all $\beta < \lambda$ we have

$$|F(\alpha,\beta) \colon \alpha \in A_\beta\}| \leq \delta.$$

Fact (AD$^+$)

Let $\lambda^+ < \Theta$ and $\kappa \leq \lambda$ a Suslin cardinal. Suppose we have the λ^+-uniform κ-covering property for $\mathcal{P}_{\omega_1}(\kappa)$. Then $\mathrm{cof}(\lambda^+) > \kappa$.

Proof: Suppose $\text{cof}(\lambda^+) \leq \kappa$ and fix $f\colon \kappa \to \lambda^+$ cofinal and continuous.

By the coding lemma and uniformization at $\boldsymbol{\Gamma} = \kappa$-Suslin, let $H\colon \omega^\omega \to \omega^\omega$ be such that for all x coding $\alpha < \kappa$ (via some $\boldsymbol{\Gamma}$-universal set P), $H(x)$ codes a one-to-one map from $[f(|x|, f(|x|+1))$ to λ.

Let G be the Kechris-Woodin generic coding function. So, for $\alpha < \kappa$ and $s \in \kappa^\omega$, $x = G(\alpha^\frown s) \in P$ with $|x| \leq \alpha$, and if $\alpha^\frown s$ enumerates an honest set $D \in \mathcal{P}_{\omega_1}(\kappa)$ then $|x| = \alpha$.

For $\beta < \lambda^+$, let $\alpha_\beta < \kappa$ be such that $\beta \in [f(\alpha_\beta), f(\alpha_\beta + 1))$.

For $S \in \mathcal{P}_{\omega_1}(\kappa)$, $\alpha < \kappa$, $p \in \kappa^{<\omega}$, and $\beta < \lambda^+$, let $F(S, p, \beta) = \gamma < \lambda$ iff

$$\forall^*_{\alpha_\beta \frown p} s \in S^\omega \; H(G(\alpha_\beta \frown s))(\beta) = \gamma$$

and otherwise set $F(S, p, \beta) = 0$.

From the λ^+-uniform κ covering property, let $A_{p,\beta} \subseteq \mathcal{P}_{\omega_1}(\kappa)$ be measure one sets such that for all $p \in \kappa < \omega$ and $\beta < \lambda^+$, $|\{F(S, p, \beta) \colon S \in A_{p,\beta}\}| \leq \kappa$.

Let $B_{p,\beta} = \{F(S, p, \beta) \colon S \in A_{p,\beta}\}$, so $B_{p,\beta} \subseteq \lambda$ and $|B_{p,\beta}| \leq \kappa$.

From the boldface GCH at λ, $|\{B_{p,\beta}\colon p \in \kappa^{<\omega}, \beta < \lambda^+\}| \leq \lambda$.

For $\beta < \lambda^+$, let $B_\beta = \{\langle p, \gamma\rangle \colon \gamma \in B_{p,\beta}\}$. So, $\{B_\beta \colon \beta < \lambda^+\}$ has size at most λ from the boldface GCH.

Let $B'_\beta = B_\beta \cup \{\langle p, \eta\rangle \colon \eta_{p\beta} < j_\mu(\kappa)\}$ where $\eta_{p\beta}$ is represented by $k(S) = \zeta$ where $F(S, p, \beta)$ is the ζth element of $A_{p,\beta}$.

However, the map $\beta \mapsto \langle \alpha_\beta, B'_\beta\rangle$ is one-to-one, a contradiction.

Supose $\beta_1 \neq \beta_2 < \lambda^+$ and suppose $\alpha_{\beta_1} = \alpha_{\beta_2} = \alpha$ and $B'_{\beta_1} = B'_{\beta_2}$.

Then for each $p \in \kappa^{<\omega}$, $A_{p,\beta_1} = A_{p,\beta_2}$.

By additivity of category and normality of μ, there is a $p \in \kappa^{<\omega}$ and functions ℓ_1,, ℓ_2 such that

$$\forall^*_\mu S \in \mathcal{P}_{\omega_1}(\kappa)\ \forall^*_p s \in S^\omega\ H(G(\alpha^\frown s))(\beta_1) = \ell_1(S),$$

and likewise for ℓ_2.

We have $\forall^*_\mu S\ F(p, S, \beta_1) = \ell_1(S)$ and $\forall^*_\mu S\ F(p, S, \beta_2) = \ell_2(S)$.

Since $B'_{\beta_1} = B'_{\beta_2}$, for almost all S we have that $F(p, S, \beta_1) = F(p, S, \beta_2)$. This contradicts $H(G(\alpha^\frown s))$ being one-to-one.

We sketch the proof of the theorem:

Theorem

Let κ be a regular limit Suslin cardinal. Let μ be a measure on κ. then there are cofinally many $\lambda^+ < j_\mu(\kappa)$ where $\mathrm{cof}(\lambda) = \omega$, and λ^+ has the stated exponent partition property (exponent $< \omega_1$ in general and κ if μ is reasonable normal).

Let $\mathbf{\Gamma}$ be the $\mathbf{\Pi}^1_1$-like scaled pointclass at κ, with scale $\{\varphi_n\}$. We write $|x|$ for $\varphi_0(x)$.

Let $C \subseteq \kappa$ be a fixed sufficiently closed c.u.b. set: for any $\alpha < \kappa$, $P_{\leq\alpha} = \{x \in P \colon |x| \leq \alpha\}$ is in $\mathbf{\Delta}$ and $\sup\{\varphi_n(x) \colon x \in P_{\leq\alpha}\} < N_C(\alpha)$. We will need to further thin C out below.

Fix a function $g: \kappa \to \kappa$ with $g(\alpha)$ of uniform cofinality ω and taking values in C'. Let $\lambda = [g]_\mu$. We show λ^+ has the stated partition property.

Two main steps:

- Show that $[\alpha \mapsto g(\alpha)^+]_\mu = \lambda^+$.
- Show that $\alpha \mapsto g(\alpha)^+$ has a block polarized partition property.

To show $[\alpha \mapsto g(\alpha)^+]_\mu = \lambda^+$ we define a "Kunen tree" K to show $[\alpha \mapsto g(\alpha)^+]_\mu \leq \lambda^+$. The lower-bound follows from the second part (a special case of it).

We define the Kunen tree K. u will code g and v will code f. This is done by the uniform coding lemma.

$(u, v, x, \vec{\alpha}, z, \vec{\beta}, w, \vec{\gamma}, y, \vec{\delta}) \in [K]$ iff

- $\vec{\alpha}$ verifies $x \in P$ and $|x| \leq \alpha_0$.
- $\vec{\beta}$ verifies $z = g_u(|x|)$.
- $\vec{\gamma}$ verifies $w = f_v(|x|)$. w does this by coding a $\Sigma_0^{g(\alpha)}$ wellfounded relation.
- y codes $(y)_i$ and $\vec{\delta}$ verifies for all i that $P(w, (y)_i, (y)_{i+1})$ and $\exists n\, |(w, y_i, y_{i+1})| \leq |(z)_n|$.

Here z codes z_n with the intention that $\sup |z_n| = f(|x|)$.

Let $U_1(\leq, <, u, z)$ the universal $\boldsymbol{\Sigma}_1(\leq, <)$ relation. The statement $U_1(\leq_x, <_x, u, z)$ is a combination of $\boldsymbol{\Sigma}^1_1$ statement about the parameters and statements $a \leq b \leq x$, $a < b \leq x$.

By $z = g_u(|x|)$ we mean $U_1(\leq_x, <_x, u, z)$.

We thin our original C.

Define $A_{\alpha,\beta,n}(x,u,w,y,y')$ iff

$$|x| \leq \alpha \wedge \exists z\, [U_1(\leq_x, <_x, u, z) \wedge |z_n| \leq \beta \wedge P^{|z_n|}(w,y,y')]$$

Then $A_{\alpha,\beta,n} \in \boldsymbol{\Delta}$.

Let $h(\alpha,\beta,n) = \sup\{\psi_m(x,u,w,y,y') \colon A_{\alpha,\beta,n}(x,u,w,y,y')\}$.

Here $\vec{\psi}$ is a scale on

$$R(x,u,w,y,y') \leftrightarrow (x \in P) \wedge \exists z\, [U_1(\leq_x, <_x, w, y, y') \wedge P^{|z_n|}(w,y,y')]$$

Take C to be closed under h.

We show the block polarized partition relation at κ, $g\colon \kappa \to \kappa$, and $h(\alpha) < g(\alpha)^+$ as above.

Let K be the Kunen tree, and $a \in \omega^\omega$ such that K_a is wellfounded and $|K_a \restriction \alpha| > h(\alpha)$ for all α.

Let $\mathcal{P}$ be a partition of the functions f of "type h." We can identify $\mathrm{dom}(f)$ with $\{(\alpha,\beta)\colon \alpha < \kappa, \beta < g(\alpha)\}$.

We use a "two-fold" application of the coding lemma to code such h, once at $\alpha < \kappa$ and once at $g(\alpha)$.

We let ν_α be the supercompactness measure on $\mathcal{P}_{\omega_1}(\alpha')$, where α' is the next reliable ordinal $\geq \alpha$.

Let $\tilde{U}_1(\leq_x, <_x, a, b)$ be a uniformization of $U_1(\leq_x, <_x, a, b)$. For $|x| = \alpha$, $\{(a, b) \colon \tilde{U}_1(\leq_x, <_x, a, b)\}$ is projective over α'-Suslin

For $\alpha < \kappa$ and $\beta < g(\alpha)$, say $u \in \omega^\omega$ is (α, β)-good if:

$$\forall |x| = \alpha \; [\exists v \; \tilde{U}_1(\leq_x, <_x, u, v) \wedge \forall |y| = \beta \; \exists w \; \tilde{U}_1(y, v, w)]$$

For $\alpha < \kappa$, $\beta < g(\alpha)$, and $\gamma < g(\alpha)^+$, let:

$$R_{\alpha,\beta,\gamma}(u) \leftrightarrow u \text{ is } (\alpha,\beta)-\text{good } \wedge \forall^* S \in \mathcal{P}_{\omega_1}(\alpha') \; \forall^* s \in S^\omega \text{ (let } x = G(s))$$
$$\forall^* T \in \mathcal{P}_{\omega_1}(\beta') \; \forall^* t \in T^\omega \text{ (let } y = G(t))$$
$$\exists v \; \exists w \; [\tilde{U}_1(\leq_x, <_x, u, v) \wedge \tilde{U}_1(\leq_y, <_y, v, w) \wedge |K_w \restriction g(\alpha)| < \gamma]$$

Then $R_{\alpha,\beta,\gamma} \in \mathbf{\Sigma}_1^{g(\alpha)}$ and the Martin conditions are satisfied.