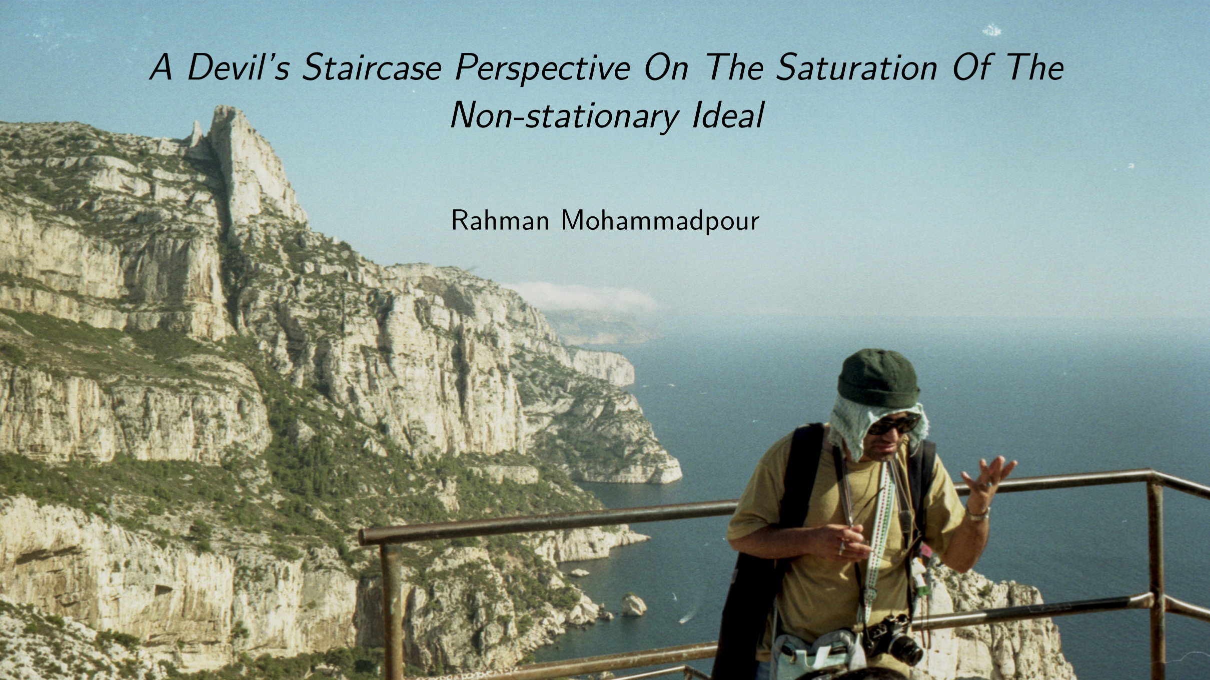
Come, let's strew blossoms and pour wine into the chalice,
Let's rend the sky's canopy and weave a novel tapestry

Hāfez (1325–1390)
Translated by ChatGPT under my supervision

A Devil's Staircase Perspective On The Saturation Of The Non-stationary Ideal

Rahman Mohammadpour



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What is the problem of saturation?

A saturation problem is a tiling problem in infinitary combinatorics, which concerns the possible number of non-negligible pieces needed to cover a set where negligible overlaps are allowed?

Ideal

Let X be a non-empty set.

An ideal $\mathcal{I}$ on X

- $\mathcal{I} \subseteq \mathcal{P}(X)$,
- $\varnothing \in \mathcal{I}$ and $X \notin \mathcal{I}$,
- if $A \subseteq B \in \mathcal{I}$, then $A \in \mathcal{I}$, and
- $\mathcal{I}$ is closed under finite unions.

An $\mathcal{I}$-positive set A

is a subset of X which is not in $\mathcal{I}$.

Saturation

Let $\mathcal{I}$ be an ideal on X.

$\mathcal{I}$ is λ-saturated if

for every covering $\langle A_\alpha : \alpha < \kappa \rangle$ of X either

- there is $\alpha < \lambda$ with $A_\alpha \in \mathcal{I}$, or
- there are $\alpha < \beta < \lambda$ with $A_\alpha \cap A_\beta \in \mathcal{I}^+$.

Equivalently, if $(\mathcal{I}^+, \subseteq^*)$ has the λ-chain condition.

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$\mathcal{I}$ is saturated if

it is $|X|^+$-saturated.

Notation

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SAT($\mathcal{I}$) states that $\mathcal{I}$ is saturated.

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Notation

- $S^\kappa_\lambda := \{\alpha < \lambda : \mathrm{cof}(\alpha) = \kappa\}$,
- NS_λ is the non-stationary ideal on λ, and
- $\mathrm{NS}^\kappa_\lambda$ is the non-stationary ideal on S^κ_λ, i.e. $\mathrm{NS}_\lambda \restriction \mathrm{S}^\kappa_\lambda$.

The problem of my perspective!!!

Open problem

Is it consistent that $\mathrm{NS}_{\omega_2}^{\omega_1}$ is saturated?

The problem of my perspective!!!

Open problem

Is it consistent that $\mathrm{NS}_{\omega_2}^{\omega_1}$ is saturated?

Question

Why is it important to answer the above question?

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Question

Why is it important to answer the above question?

Answer

No pain, no gain!

Those were the days...

Let λ be a regular uncountable cardinal.

[1]"Real-valued measurable cardinals" 1971.

[2]"On the closed unbounded ideal of ordinal numbers" 1973.

[3]"On splitting stationary subsets of large cardinals" 1977.

[4]Optimal: Jech and Woodin,"Saturation of the closed unbounded filter on the set of regular cardinals" 1985

[5]"The saturation of a product of ideals" 1980.

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- (Solovay[1]) Every stationary set $S \subseteq \lambda$ can be decomposed into λ many stationary sets. (i.e., $\mathrm{NS}_\lambda \restriction \mathrm{S}$ is not λ-saturated)

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- (Wagon[5]) $\mathrm{NS}_\lambda \ltimes \mathrm{NS}_\lambda$ is *nowhere saturated*.

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Gitik and Shelah's seminal contribution[6]

Building on the earlier independent and joint works by Gitik, Shelah, Džamonja and Shelah and others, Gitik and Shelah proved:

- $\mathrm{NS}^{\mathrm{cof}(\lambda)}_{\lambda^+}$ is not λ^{++}-saturated for singular λ.
- NS_λ is not saturated for weakly inaccessible λ.
- $\mathrm{NS}^\kappa_\lambda$ is not saturated for a regular κ below a weakly inaccessible λ.

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Theorem (Gitik–Shelah)

For regular cardinals $\kappa < \lambda$ with $\kappa^+ < \lambda$,

1. *ZFC $\vdash \neg$SAT(NS_λ).*
2. *ZFC $\vdash \neg$SAT($\mathrm{NS}^\kappa_\lambda$).*

[6] "Less saturated ideals" 1997.

Some consistency results

Let λ be a regular uncountable cardinal.

- (Kunen[7]) It is consistent there to have a saturated ideal on λ^+.
- (Woodin[8]) It is consistent that $\mathrm{NS}^{\lambda}_{\lambda^+}$ is *locally saturated*, i.e., $\mathrm{NS}_{\lambda^+} \upharpoonright \mathrm{S}$ is saturated for some stationary set $S \subseteq S^{\lambda}_{\lambda^+}$.

[7] "Saturated ideals" 1978.

[8] Archived: see Foreman and Komjath, "The club guessing ideal: commentary on a theorem of Gitik and Shelah" 2005

So far

Let λ be a regular uncountable cardinal.

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- We have not discussed $\mathrm{NS}^{\lambda}_{\lambda^+}$.
- There are more results that even I will not go into!

NS_{ω_1}

- (Steel–Van Wesep[9]) ZFC + SAT(NS_{ω_1}) is consistent relative to ZF + $\mathsf{AD}_{\mathbb{R}}$ + Θ is regular.

[9] "Two consequences of determinacy consistent with choice" 1982.
[10] "Some consistency results in ZFC using AD" 1983.
[11] "Martin's maximum, saturated ideals, and nonregular ultrafilters I" 1988.
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- (Foreman–Magidor–Shelah[11]) MM implies SAT(NS_{ω_1}),
- (Shelah[12]) SAT(NS_{ω_1}) is consistent relative to a Woodin cardinal.

---

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Pop-up question

Why is it important to answer the above question?

The leftover

This is the problem:

Is it consistent that $\mathrm{NS}_{\omega_2}^{\omega_1}$ is saturated?

Pop-up question

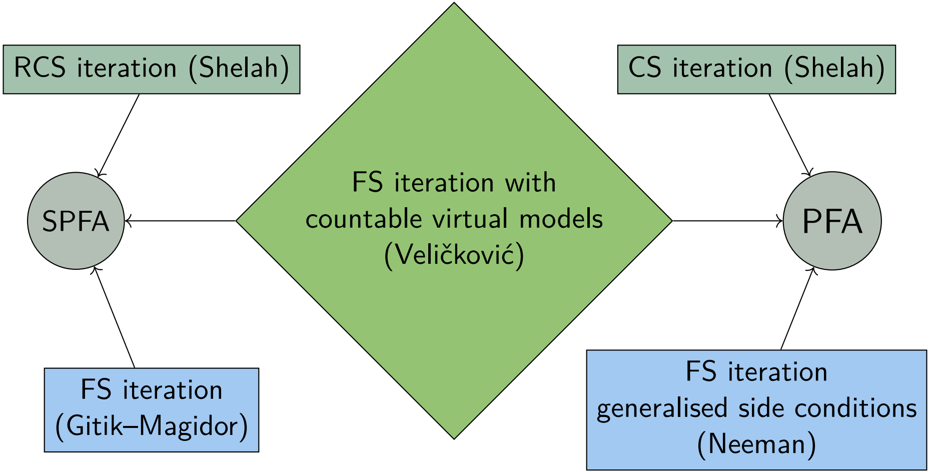
Why is it important to answer the above question?

The answer

This is a wrong question!

Iteration and forcing axioms

Iteration and forcing axioms



The status of strong higher forcing axioms

Blindengarten, Bonn, Germany

Some evidence[13]

M.–Veličković

Starting from two supercompact cardinals, it is consistent to have stationarily many elementary submodels of size $\aleph_2$, which are the union of a ω_1-continuous $\in$-sequence of $\aleph_1$-guessing models of size $\aleph_1$.

The above conclusion

- implies $2^{\aleph_0} \geq \aleph_3$.
- is a consequence of an imaginary strong higher forcing axiom.

The proof

- uses "finite conditions", and
- preserves three cardinals ω_1 and two supercompact cardinals by using sequences of models of three types.

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[13] "Guessing models and the approachability ideal" 2021.

To tackle SAT($\mathrm{NS}^{\omega_1}_{\omega_2}$)

- The idea is to use the method of side conditions developed by B. Veličković and the speaker.

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- The main problem is that the natural corresponding iterable class is too small.
- We do not know how to iterate the forcings relevant for sealing antichains.

To tackle SAT($\mathrm{NS}^{\omega_1}_{\omega_2}$)

- The idea is to use the method of side conditions developed by B. Veličković and the speaker.
- The main problem is that the natural corresponding iterable class is too small.
- We do not know how to iterate the forcings relevant for sealing antichains.
- But we are not giving up!

Natural questions:

1. What is the exact strategy?
2. What forces us to assume that the method is relevant?
3. What is the large cardinal assumption?
4. Does the same strategy also apply to the known case of NS?

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1. What is the exact strategy?
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Be less ambitious, Rahman, and examine the well-known case of NS_{ω_1} using Veličković's iteration of semi-proper forcings.

SAT(NS_{ω_1}) under MM using original proof

- The original proof of MM $\Rightarrow$ SAT(NS_{ω_1}) by sealing an antichain:
  - $\mathcal{A}$ a maximal antichain.
  - collapse the size of $\mathcal{A}$ to be ω_1.
  - and then shoot a Baumgartner club into $\nabla\mathcal{A}$.
  - Recover the desirable object by MM.

SAT(NS_{ω_1}) under MM using finite conditions

Let $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$ be maximal. Set $\mathcal{S}_\theta(\mathcal{A}) = \{M \prec \mathcal{H}_\theta : \exists\alpha \in M \cap \kappa, \delta_M \in A_\alpha\}$.

- $\mathcal{S}_\theta(\mathcal{A})$ is projective stationary as the antichain $\mathcal{A}$ is maximal.

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- $\mathcal{S}_\theta(\mathcal{A})$ is projective stationary as the antichain $\mathcal{A}$ is maximal.
- Let the forcing $\mathbb{P}^1_{\mathcal{A}}$ consist of pairs $p = (\mathcal{M}_p, d_p)$, where
  1. (side conditions) $\mathcal{M}_p \subseteq \mathcal{S}_\theta(\mathcal{A})$ is a finite $\in$-chain, and
  2. (decorations) $d_p : \mathcal{M}_p \to \mathcal{P}_\omega(H_\theta)$ is a function so that if $M \in N$, then $d_p(M) \in N$.

  $p \leq q$ if and only if $\mathcal{M}_p \supseteq \mathcal{M}_q$ and that for every $M \in \mathcal{M}_q$, $d_p(M) \supseteq d_q(M)$.

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- $\mathbb{P}^1_{\mathcal{A}}$ is strongly proper for $\mathcal{S}_\theta(\mathcal{A})$.

SAT(NS_{ω_1}) by RCS iteration

- RCS iteration up to a Woodin cardinal, where
- at stage α force the sealing poset if semi-proper,
- otherwise, collapse $2^{\aleph_2}$ onto ω_1 with countable conditions.

SAT(NS_{ω_1}) *by RCS iteration*

- RCS iteration up to a Woodin cardinal, where
- at stage α force the sealing poset if semi-proper,
- otherwise, collapse $2^{\aleph_2}$ onto ω_1 with countable conditions.

- More or less the same with Veličković's iteration!

Sealing an antichain in $\mathrm{NS}_{\omega_2}^{\omega_1}$

Similarly,

Proposition

Assume $\mathcal{A}$ is a maximal antichain in $\mathrm{NS}_{\omega_2}^{\omega_1}$. Then there is a proper forcing $\mathbb{P}_{\mathcal{A}}^2$ with finite conditions, which is semi-proper for a $S_{\omega_2}^{\omega_1}$-projective stationary set such that it seals $\mathcal{A}$.

The map of Veličković's iteration

The construction has the following important components.

- $\mathbb{P}_\alpha$, the iteration up to α.
- $\mathbb{M}(\mathbb{P}_\alpha)$, the scaffolding of $\mathbb{P}_\alpha$.
- $\mathbb{P}_{\mathrm{next}(\alpha)} \sim \mathbb{M}(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha)$, the $\mathbb{P}_\alpha$-scaffolding poset of $\dot{\mathbb{Q}}_\alpha$. It replaces $\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$.

• Note: The initial segment $\mathbb{P}_\alpha$ of the iteration is inductively obtained as in the above manner.

The important ingredient

Scaffolding poset for semi-proper forcings: Assume $\mathbb{P}$ is a semi-proper forcing. Let $\mathbb{M}^1(\mathbb{P})$ consist of $p = (\mathcal{M}_p, w_p)$ such that:

1. $\mathcal{M}_p$ is a finite collection of models,
2. $w_p \in \mathbb{P}$ is semi-generic for every model in $\mathcal{M}_p$,
3. w_p forces $\mathcal{M}_p$ to form a favourable chain!

The order is natural!

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The order is natural!

Fact

$\mathbb{M}^1(\mathbb{P})$ is semi-proper.

Scaffolding poset vs canonical genericity

A condition $p \in \mathbb{P}^1_{\mathcal{A}}$ is canonically semi-generic for a countable model M if there is $M' \in \mathcal{M}_p$ which end-extends M.

Scaffolding poset vs canonical genericity

A condition $p \in \mathbb{P}^1_{\mathcal{A}}$ is canonically semi-generic for a countable model M if there is $M' \in \mathcal{M}_p$ which end-extends M. Now the crane poset: $\mathbb{M}^1[\mathbb{P}^1_{\mathcal{A}}]$ consists of $p = (\mathcal{M}_p, \mathcal{M}'_p, d_p)$ such that

- $\mathcal{M}_p$ is a finite collection of models,
- $(\mathcal{M}'_p, d_p)$ is canonically semi-generic for every $M \in \mathcal{M}_p$,
- $(\mathcal{M}'_p, d_p)$ forces $\mathcal{M}_p$ to be a favourable chain.

The order is natural.

Proposition

$\mathbb{M}^1[\mathbb{P}^1_{\mathcal{A}}]$ is stationary set-preserving forcing.

Conclusion

While we do not know how to iterate $\mathbb{P}^2_{\mathcal{A}}$, even that how to make a scaffolding poset for $\mathbb{P}^2_{\mathcal{A}}$ in Veličković's style, we have a hint how to use the canonical semi-generic conditions for models of size $\aleph_1$. We can form $\mathbb{M}^2[\mathbb{P}^2_{\mathcal{A}}]$.

Conclusion

While we do not know how to iterate $\mathbb{P}^2_{\mathcal{A}}$, even that how to make a scaffolding poset for $\mathbb{P}^2_{\mathcal{A}}$ in Veličković's style, we have a hint how to use the canonical semi-generic conditions for models of size $\aleph_1$. We can form $\mathbb{M}^2[\mathbb{P}^2_{\mathcal{A}}]$. The trial case is answering the following question, which is done partially.

Problem

Assume a Woodin cardinal. Is it possible to force NS_{ω_1} to be saturated with finite conditions in an essential way?

"Where did I come from, and what am I supposed to be doing?
I have no idea. My soul is from elsewhere, I'm sure of that,
and I intend to end up there."

Rumi, translated by Coleman Barks

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The perspective presented has been co-funded by the European Commission and the Polish National Science Centre under the Marie Skłodowska-Curie COFUND grant as a project entitled "Side Conditions and the Saturation of the Non-stationary Ideal"" with the speaker as P.I. and Grigor Sargsyan as mentor. The project will start one day next spring at the IMPAN branch in Gdansk.

The speaker is currently supported by the Austrian Academy of Sciences through the APART-MINT fellowship.