Set theory and derived limits

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Introduction

This talk will be about some interactions between set theory and homological algebra, particularly around derived functors of the inverse limit functor. This interaction has been fruitful in both directions:

- Questions coming from homological algebra have led to the development of new methods and notions in set theory.
- Set theoretic techniques have allowed for the solution of various open problems in algebra.

The talk is about the functor $\varprojlim$, which takes inverse systems of abelian groups to abelian groups. In general, this functor fails to be exact, i.e., fails to preserve short exact sequences. Given $n > 0$, the derived functor $\varprojlim^n$ measures the "n-dimensional" obstructions to the exactness of $\varprojlim$.

Question

Given an $n > 0$ and an inverse system $\mathbf{A}$ of abelian groups, when is it the case that $\varprojlim^n \mathbf{A} = 0$?

A brief history of $\varprojlim$

The cross-pollination between set theory and the study of derived limits has been most active during three distinct historical periods:

1. **1965–1975**
   - (Goblot, Mitchell) connections between the (non)vanishing of $\varprojlim^n$ and the cofinality of the directed system in question
   - (Osofsky) connections between the projective dimension of certain rings/fields and the value of the continuum
2. **1988–1997**
   - intensive study of the *first* derived limit of a particular family of inverse systems indexed by ${}^\omega\omega$ that arose naturally in the study of strong homology
   - connections with familiar topics in set theory (cardinal characteristics, PFA, OCA, forcing, . . .)
3. **2015–now**
   - development of new tools for understanding *higher* derived limits (i.e., $\varprojlim^n$ for $n > 1$);
   - further applications to pure set theory, strong homology, and condensed mathematics

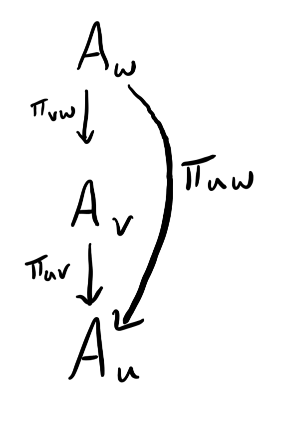
I. Inverse systems and limits

Inverse systems

Definition

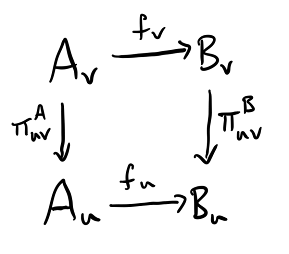
Suppose that $(\Lambda, \leq)$ is a directed set. An *inverse system* (of abelian groups) indexed by Λ is a family $\mathbf{A} = \langle A_u,\ \pi_{uv} \mid u \leq v \in \Lambda \rangle$ such that:

- for all $u \in \Lambda$, A_u is an abelian group;
- for all $u \leq v \in \Lambda$, $\pi_{uv} : A_v \to A_u$ is a group homomorphism;
- for all $u \leq v \leq w \in \Lambda$, $\pi_{uw} = \pi_{uv} \circ \pi_{vw}$.



Level morphisms

If **A** and **B** are two inverse systems indexed by the same directed set, Λ, then a *level morphism* from **A** to **B** is a family of group homomorphisms $\mathbf{f} = \langle f_u : A_u \to B_u \mid u \in \Lambda \rangle$ such that, for all $u \leq v \in \Lambda$, $\pi^B_{uv} \circ f_v = f_u \circ \pi^A_{uv}$.



With this notion of morphism, the class of all inverse systems indexed by a fixed directed set Λ becomes a well-behaved category $\mathrm{Ab}^{\Lambda^{\mathrm{op}}}$ (in particular, it is an abelian category).

Inverse limits

If $\mathbf{A}$ is an inverse system indexed by Λ, then we can form the *inverse limit*, $\varprojlim \mathbf{A}$ (or just $\lim A$), which is itself an abelian group. Concretely, $\lim \mathbf{A}$ can be seen as the subgroup of $\prod_{u \in \Lambda} A_u$ consisting of all sequences $\langle a_u \mid u \in \Lambda \rangle$ such that, for all $u \leq v \in \Lambda$, we have $a_u = \pi_{uv}(a_v)$.

If $\mathbf{A}$ and $\mathbf{B}$ are inverse systems and $\mathbf{f} : \mathbf{A} \to \mathbf{B}$, then $\mathbf{f}$ lifts to a group homomorphism $\lim \mathbf{f} : \lim \mathbf{A} \to \lim \mathbf{B}$. Concretely, this is done by letting $\lim \mathbf{f}(\langle a_u \mid u \in \Lambda \rangle) = \langle f_u(a_u) \mid u \in \Lambda \rangle$ for all $\langle a_u \mid u \in \Lambda \rangle \in \lim \mathbf{A}$.

This turns lim into a *functor* from the category $\mathrm{Ab}^{\Lambda^{\mathrm{op}}}$ of inverse systems indexed by Λ to the category Ab of abelian groups.

Question: How “nice” is this functor?

Exact sequences

In the category of inverse systems, kernels, images, and quotients can be defined pointwise in the obvious way. For example, if $\mathbf{f} : \mathbf{A} \to \mathbf{B}$ is a level morphism, then $\ker(\mathbf{f})$ can be seen as the inverse system $\langle \ker(f_u), \pi_{uv} \mid u \leq v \in \Lambda \rangle$, where π_{uv} is simply $\pi^{A}_{uv} \restriction \ker(f_v)$.

We say that a pair of morphisms $\mathbf{A} \xrightarrow{\mathbf{f}} \mathbf{B} \xrightarrow{\mathbf{g}} \mathbf{C}$ is *exact at* $\mathbf{B}$ if $\mathrm{im}(\mathbf{f}) = \ker(\mathbf{g})$. A *short exact sequence* is a sequence $\mathbf{0} \to \mathbf{A} \xrightarrow{\mathbf{f}} \mathbf{B} \xrightarrow{\mathbf{g}} \mathbf{C} \to \mathbf{0}$ that is exact at $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$.

In a short exact sequence as above, we have $\ker(\mathbf{f}) = \mathbf{0}$ ($\mathbf{f}$ is *injective*) and $\mathrm{im}(\mathbf{g}) = \mathbf{C}$ ($\mathbf{g}$ is *surjective*). It can be helpful to think of $\mathbf{A}$ as a *subobject* of $\mathbf{B}$ and to think of $\mathbf{C}$ as the quotient $\mathbf{B}/\mathbf{A}$.

Exact functors

A functor F between abelian categories is said to be *exact* if it preserves short exact sequences, i.e., if, whenever $\mathbf{0} \to \mathbf{A} \xrightarrow{\mathbf{f}} \mathbf{B} \xrightarrow{\mathbf{g}} \mathbf{C} \to \mathbf{0}$ is exact in the source category of F, $\mathbf{0} \to F\mathbf{A} \xrightarrow{F\mathbf{f}} F\mathbf{B} \xrightarrow{F\mathbf{g}} F\mathbf{C} \to \mathbf{0}$ is exact in the target category of F.

The inverse limit functor is *left exact*: if $\mathbf{0} \to \mathbf{A} \xrightarrow{\mathbf{f}} \mathbf{B} \xrightarrow{\mathbf{g}} \mathbf{C}$ is exact at $\mathbf{A}$ and $\mathbf{B}$, then $\mathbf{0} \to \lim \mathbf{A} \xrightarrow{\lim \mathbf{f}} \lim \mathbf{B} \xrightarrow{\lim \mathbf{g}} \lim \mathbf{C}$ is exact at $\lim \mathbf{A}$ and $\lim \mathbf{B}$. However, it fails to be exact, i.e., even if $\mathrm{im}(\mathbf{g}) = \mathbf{C}$, we might have $\mathrm{im}(\lim \mathbf{g}) \neq \lim \mathbf{C}$.

The failure of lim to be exact essentially amounts to the failure of lim to preserve quotients: if the quotient system $\mathbf{B}/\mathbf{A}$ is defined, then it need not be the case that $\lim \mathbf{B}/\mathbf{A} \cong \lim \mathbf{B}/\lim \mathbf{A}$.

An example ($\Lambda = \omega$)

$$\mathbf{0} \longrightarrow \mathbf{A} \xrightarrow{\mathbf{f}} \mathbf{B} \xrightarrow{\mathbf{g}} \mathbf{C} \longrightarrow \mathbf{0}$$

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow{\scriptstyle \times 2} & & \downarrow{\scriptstyle \times 2} & & \downarrow{\scriptstyle \times 2} & & \downarrow \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 3} & \mathbb{Z} & \xrightarrow{\text{mod } 3} & \mathbb{Z}/3 & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \times 2} & & \downarrow{\scriptstyle \times 2} & & \downarrow{\scriptstyle \times 2} & & \downarrow \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 3} & \mathbb{Z} & \xrightarrow{\text{mod } 3} & \mathbb{Z}/3 & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \times 2} & & \downarrow{\scriptstyle \times 2} & & \downarrow{\scriptstyle \times 2} & & \downarrow \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 3} & \mathbb{Z} & \xrightarrow{\text{mod } 3} & \mathbb{Z}/3 & \longrightarrow & 0
\end{array}$$

$\lim \mathbf{A} = \lim \mathbf{B} = 0$ and $\lim \mathbf{C} = \mathbb{Z}/3$, so applying lim to this short exact sequence yields $0 \to 0 \to 0 \to \mathbb{Z}/3 \to 0$, which is not exact at $\mathbb{Z}/3$.

Derived functors

Given any left exact functor F, there is a general procedure for producing a sequence of (right) derived functors $\langle F^n \mid n \in \omega \setminus \{0\} \rangle$ that "measure" the failure of the functor F to be exact. These derived functors then take short exact sequences

$$\mathbf{0} \longrightarrow \mathbf{A} \xrightarrow{\mathbf{f}} \mathbf{B} \xrightarrow{\mathbf{g}} \mathbf{C} \longrightarrow \mathbf{0}$$

to *long* exact sequences

$$\mathbf{0} \longrightarrow F\mathbf{A} \xrightarrow{F\mathbf{f}} F\mathbf{B} \xrightarrow{F\mathbf{g}} F\mathbf{C} \xrightarrow{\delta} F^1\mathbf{A} \xrightarrow{F^1\mathbf{f}} F^1\mathbf{B} \xrightarrow{F^1\mathbf{g}} F^1\mathbf{C} \xrightarrow{\delta} F^2\mathbf{A} \xrightarrow{F^2\mathbf{f}} F^2\mathbf{B} \xrightarrow{F^2\mathbf{g}} F^2\mathbf{C} \longrightarrow \dots$$

We will be interested in the derived functors $\langle \lim^n \mid n \in \omega \setminus \{0\} \rangle$.

Derived limits and cofinality

A pair of complementary theorems from the early 1970s demonstrates a connection between the vanishing of derived inverse limits and the cofinality of the indexing poset.

Theorem (Goblot, 1970 [7])

Suppose that Λ *is a directed set,* $n < \omega$*, and* $\mathrm{cf}(\Lambda) \leq \aleph_n$*. Then, for every* $\mathbf{A} \in \mathsf{Ab}^{\Lambda^{\mathrm{op}}}$*, we have*

$$\lim{}^{n+2}\mathbf{A} = 0.$$

Theorem (B. Mitchell, 1973 [10])

Suppose that Λ *is a directed set,* $n < \omega$*, and* $\mathrm{cf}(\Lambda) \geq \aleph_n$*. Then there is* $\mathbf{A} \in \mathsf{Ab}^{\Lambda^{\mathrm{op}}}$ *such that*

$$\lim{}^{n+1}\mathbf{A} \neq 0.$$

Homological dimension

Theorem (Osofsky, 1970 [11])

Let $\{F_i \mid i < \omega\}$ be a family of fields. Then the global dimension of $\prod_\omega F_i$ is $n+1$ iff $2^{\aleph_0} = \aleph_n$ (and is infinite if $2^{\aleph_0} > \aleph_\omega$).

> *In this paper, as well as in [–], statements on homological dimension were found to be equivalent to the continuum hypothesis. In these works, if $2^{\aleph_0} \neq \aleph_1$, then $\aleph_1$ appears in the role of a stumbling block in getting from $\aleph_0$ to $2^{\aleph_0}$ . . . There is no way in these papers to get one's hands on $\aleph_1$. Such a situation is aesthetically (or intuitively, if you prefer) repugnant to me . . . For those reasons, the hypothesis $2^{\aleph_0} = \aleph_1$ appears to me to be the natural one applying to the axiom system in which homological algebra is done, and $2^{\aleph_0} > \aleph_\omega$ has somewhat upsetting consequences.*

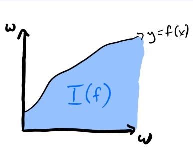
Barbara Osofsky, "Homological dimension and cardinality", 1970

II. The first derived limit

The system $\mathbf{A}[H]$

Fix an abelian group H. Given a function $f : \omega \to \omega$, let

$$I(f) := \{(k, m) \in \omega \times \omega \mid m \leq f(k)\}$$



and let $A_f[H] := \bigoplus_{I(f)} H$. Given $f, g \in {}^{\omega}\omega$, let $f \leq g$ iff $f(k) \leq g(k)$ for all $k < \omega$; in this case, let $\pi_{fg} : A_g[H] \to A_f[H]$ be the projection map. This defines an inverse system

$$\mathbf{A}[H] = \langle A_f, \pi_{fg} \mid f, g \in {}^{\omega}\omega,\ f \leq g \rangle.$$

If $H = \mathbb{Z}$, we omit it in the notation. Note that

$$\lim \mathbf{A}[H] = \bigoplus_{\omega} \prod_{\omega} H.$$

Strong homology

The system **A** naturally arises in the study of the additivity of strong homology, a homology theory for topological spaces that is strong shape invariant. Strong homology was developed by Lisica and Mardešić, and was designed to reflect the properties of spaces with pathological local behavior more reliably than, e.g., singular homology.

Given a space X and a $p < \omega$, let $\bar{H}_p(X)$ denote the p^{th} strong homology group of X.

Additivity of strong homology

A desirable property for a homology theory to have is *additivity*:

Definition

A homology theory is *additive* on a class of topological spaces $\mathcal{C}$ if, for every natural number p and every family $\{X_i \mid i \in J\}$ such that each X_i and $\coprod_J X_i$ are in $\mathcal{C}$, we have

$$\bigoplus_J \mathrm{H}_p(X_i) \cong \mathrm{H}_p(\coprod_J X_i)$$

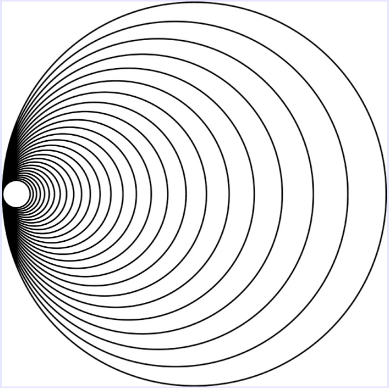
via the map induced by the inclusions

$$X_i \hookrightarrow \coprod_J X_i.$$

Question: Is strong homology additive?

Infinite earring spaces

Let X^n denote the n-dimensional infinite earring space, i.e., the one-point compactification of the disjoint union of countably infinitely many copies of the n-dimensional open unit ball.



X^1

Additivity and $\lim^n \mathbf{A}$

Theorem (Mardešić-Prasolov, '88 [9])

Suppose that $0 < p < n$ are natural numbers. Then

$$\bigoplus_{\mathbb{N}} \bar{\mathrm{H}}_p(X^n) = \bar{\mathrm{H}}_p(\coprod_{\mathbb{N}} X^n)$$

if and only if $\lim^{n-p} \mathbf{A} = 0$.

Consequently, if strong homology is additive, even on closed subsets of Euclidean space, then $\lim^n \mathbf{A} = 0$ for all $n \geq 1$.

Theorem (Mardešić–Prasolov, Simon, 1988 [9])

If CH *holds, then* $\lim^1 \mathbf{A} \neq 0$*, and hence strong homology is not additive, even on closed subspaces of Euclidean space.*

Describing $\lim^1 \mathbf{A}$

Define an inverse system $\mathbf{B} = \langle B_f, \pi_{fg} \mid f, g \in {}^{\omega}\omega,\ f \leq g \rangle$ by letting $B_f = \prod_{I(f)} \mathbb{Z}$. Note that $\lim \mathbf{B} = \prod_{\omega \times \omega} \mathbb{Z}$. This gives rise to a short exact sequence

$$0 \to \mathbf{A} \to \mathbf{B} \to \mathbf{B}/\mathbf{A} \to 0,$$

which then induces a long exact sequence

$$0 \to \lim \mathbf{A} \to \lim \mathbf{B} \to \lim \mathbf{B}/\mathbf{A} \to \lim^1 \mathbf{A} \to \lim^1 \mathbf{B} \to \dots$$

$\mathbf{B}$ has the property that $\lim^n \mathbf{B} = 0$ for all $n > 0$. Therefore, we get

$$\lim^1 \mathbf{A} \cong \frac{\lim \mathbf{B}/\mathbf{A}}{\mathrm{im}(\lim \mathbf{B})}.$$

Describing $\lim^1 \mathbf{A}$

Let $=^*$ denote equality mod finite. For $f \in {}^\omega\omega$, elements of B_f/A_f are the $=^*$-equivalence classes of functions from $I(f)$ to $\mathbb{Z}$. Therefore, elements of $\lim \mathbf{B}/\mathbf{A}$ are (equivalence classes of) families of functions

$$\langle \varphi_f : I(f) \to \mathbb{Z} \mid f \in {}^\omega\omega \rangle$$

that are *coherent*, i.e., $\varphi_f =^* \varphi_g$ (on their common domain $I(f) \cap I(g)$) for all $f, g \in {}^\omega\omega$.

Elements of $\mathrm{im}(\lim \mathbf{B})$ are precisely those coherent families of functions for which there is a single function $\psi : \omega \times \omega \to \mathbb{Z}$ such that $\psi \restriction I(f) =^* \varphi_f$ for all $f \in {}^\omega\omega$. Such families are called *trivial*.

We thus see that $\lim^1 \mathbf{A} = 0$ iff every coherent family of functions is trivial.

Some results

- (Mardešić–Prasolov, Simon, 1988 [9]) If CH holds, then $\lim^1 \mathbf{A} \neq 0$.
- (Dow–Simon–Vaughan, 1989 [6]) If $\mathfrak{d} = \aleph_1$, then $\lim^1 \mathbf{A} \neq 0$.
- (Dow–Simon–Vaughan, 1989 [6]) If the Proper Forcing Axiom holds, then $\lim^1 \mathbf{A} = 0$.
- (Todorcevic, 1989 [12]) If the Open Coloring Axiom holds, then $\lim^1 \mathbf{A} = 0$.
- (Kamo, 1993 [8]) After adding $\aleph_2$-many Cohen reals to any model of ZFC, we have $\lim^1 \mathbf{A} = 0$.

The system $\mathbf{A}_\kappa[H]$

Fix an abelian group H and a cardinal κ. Define an inverse system

$$\mathbf{A}_\kappa[H] = \langle A_\alpha, \pi_{\alpha\beta} \mid \alpha \leq \beta < \kappa \rangle$$

by letting $A_\alpha = \bigoplus_\alpha H$ and $\pi_{\alpha\beta} : A_\beta \to A_\alpha$ be the projection map for all $\alpha < \beta < \kappa$. As with $\mathbf{A}[H]$, we can see that $\lim^1 \mathbf{A}_\kappa[H] \neq 0$ iff there is a family of functions $\langle \varphi_\alpha : \alpha \to H \mid \alpha < \kappa \rangle$ that is

1. coherent: $\varphi_\alpha =^* \varphi_\beta \restriction \alpha$ for all $\alpha \leq \beta < \kappa$; and
2. nontrivial: there is no $\psi : \kappa \to H$ such that $\varphi_\alpha =^* \psi \restriction \alpha$ for all $\alpha < \kappa$.

If $|H| < \kappa$, then this is equivalent to the existence of coherent κ-Aronszajn subtree of ${}^{<\kappa}H$. In particular:

- $\lim^1 \mathbf{A}_{\omega_1} \neq 0$;
- (Todorcevic) The P-Ideal Dichotomy implies that $\lim^1 \mathbf{A}_\kappa = 0$ for all regular $\kappa > \omega_1$.

III. Higher dimensions

Higher coherence

An analogous characterization of the nonvanishing of $\lim^n \mathbf{A}$ for $n > 1$ exists in terms of higher-dimensional families of functions. For example, $\lim^2 \mathbf{A} \neq 0$ if and only if there is a family

$$\Phi = \langle \varphi_{fg} : I(f) \cap I(g) \to \mathbb{Z} \mid f, g \in {}^{\omega}\omega \rangle$$

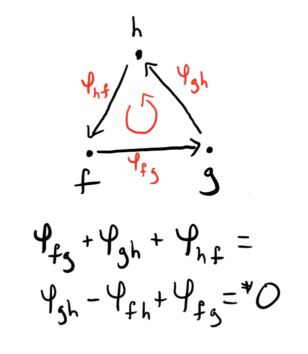
that is

- alternating: $\varphi_{fg} = -\varphi_{gf}$ for all $f, g \in {}^{\omega}\omega$;
- 2-coherent: $\varphi_{fh} =^* \varphi_{fg} + \varphi_{gh}$ for all $f, g, h \in {}^{\omega}\omega$;
- nontrivial: there is no family $\langle \psi_f : I(f) \to \mathbb{Z} \mid f \in {}^{\omega}\omega \rangle$ such that $\varphi_{fg} =^* \psi_g - \psi_f$ for all $f, g \in {}^{\omega}\omega$.

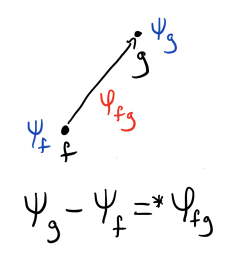
Similar characterizaitons exist for higher dimensions, and for the systems $\mathbf{A}_\kappa$. In particular, nontrivial elements of $\lim^n \mathbf{A}_\kappa$ can naturally be seen as n-dimensional analogues of coherent κ-Aronszajn trees.

A reframing

Coherence and triviality can be reframed in terms of oriented sums of functions indexed by maximal faces of simplices whose vertices are labeled by elements of ${}^{\omega}\omega$. For example, a 2-dimensional family $\langle \varphi_{fg} \mid f, g \in {}^{\omega}\omega \rangle$ is 2-coherent if the oriented sum on the boundary of every 2-simplex vanishes mod finite:



A 2-d family is trivial if its 2-d information reduces (mod finite) to a 1-d family $\langle \psi_f \mid f \in {}^{\omega}\omega \rangle$.



Nonvanishing results

The past decade has seen significant progress in our understanding of higher derived limits. We begin by surveying some recent nonvanishing results about $\lim^n \mathbf{A}$.

- (Bergfalk, 2017 [3]) PFA implies $\lim^2 \mathbf{A} \neq 0$.
- (Veličković–Vignati, 2021 [13]) For all $n \geq 1$, it is consistent that $\lim^n \mathbf{A} \neq 0$. In particular, if $\mathfrak{b} = \mathfrak{d} = \aleph_n$ and a version of weak diamond holds at $\aleph_k$ for all $1 \leq k \leq n$, then $\lim^n \mathbf{A} \neq 0$. The requirement $\mathfrak{b} = \mathfrak{d} = \aleph_n$ was recently weakened by Casarosa.
- (LH, 2023) If $\mathfrak{d} = \aleph_n$, then $\lim^n \mathbf{A}[\bigoplus_{\omega_n} \mathbb{Z}] \neq 0$.

Vanishing results

A progression of recent results has shown that we can consistently obtain simultaneous vanishing of $\lim^n \mathbf{A}$ for all $n > 0$.

- (Bergfalk–LH, 2021 [5]) After adding a weakly compact number of Hechler reals, $\lim^n \mathbf{A} = 0$ for all $n > 0$.
- (Bergfalk–Hrušák–LH, 2023 [4]) After adding $\beth_\omega$-many Cohen reals, $\lim^n \mathbf{A} = 0$ for all $n > 0$. In particular, this conclusion is consistent with $2^{\aleph_0} = \aleph_{\omega+1}$.
- (Bannister, 2023 [1]) In either of the above models, we in fact have $\lim^n \mathbf{A}[H] = 0$ for all $n > 0$ and all abelian groups H.

Questions

A number of prominent open questions remain. For example:

- Does $\mathfrak{d} = \aleph_n$ imply $\lim^n \mathbf{A} \neq 0$? More generally, if $\lim^n \mathbf{A} = 0$ for all $n > 0$, must we have $2^{\aleph_0} > \aleph_\omega$?
- How much simultaneous nonvanishing of $\lim^n \mathbf{A}$ can we have? Can we have $\lim^n \mathbf{A} \neq 0$ for infinitely many (or all) $n > 0$?
- One can define an analogous system $\mathbf{A}_{^\kappa\omega}$ indexed by $^\kappa\omega$ instead of $^\omega\omega$. It is known that, for all κ, $\lim^1 \mathbf{A}_{^\kappa\omega} = 0$ iff $\lim^1 \mathbf{A} = 0$. Is this also true of $\lim^n$ for $n > 1$?

Back to $\mathbf{A}_\kappa$

There has also been recent progress on our understanding of $\lim^n \mathbf{A}_\kappa$. Classical results already gave us some information below $\aleph_\omega$:

- (Goblot) For all $m < n < \omega$, $\lim^n \mathbf{A}_{\omega_m} = 0$.
- (Mitchell) For all $n < \omega$, $\lim^n \mathbf{A}_{\omega_n}[\bigoplus_n \mathbb{Z}] \neq 0$.

Recall also that PID (and hence PFA) implies that $\lim^1 \mathbf{A}_\kappa = 0$ for all regular $\kappa > \omega_1$. Against this backdrop, we have the following two complementary results.

- If κ is weakly compact, then $\lim^n \mathbf{A}_\kappa = 0$ for all $n > 0$. If λ is strongly compact, then $\lim^n \mathbf{A}_\kappa[H] = 0$ for all $n > 0$, all $\kappa \geq \lambda$, and all abelian groups H.
- (Bergfalk–LH) If $V = L$, then $\lim^n \mathbf{A}_\kappa \neq 0$ for all $n > 0$ and every regular $\kappa \geq \aleph_n$ that is not weakly compact.

Some vanishing results

Theorem (Bergfalk–LH–Zhang)

Relative to the consistency of a supercompact cardinal, it is consistent that $\lim^n \mathbf{A}_{\aleph_{\omega+1}}[H] = 0$ *for all* $1 \leq n < \omega$ *and all abelian groups* H*.*

This leaves open the question of the consistent vanishing of $\lim^n \mathbf{A}_{\aleph_m}$ for $1 < n < m < \omega$.

Theorem (Bergfalk–LH–Zhang)

Suppose that κ *is weakly compact. Then, in the extension by the Mitchell forcing* $\mathbb{M}(\omega_1, \kappa)$*, there are no 2-coherent, nontrivial families* $\langle \varphi_{\alpha\beta} : \alpha \to \mathbb{Z} \mid \alpha < \beta < \omega_3 \rangle$ *of countably supported functions.*

If $V = L$, the families witnessing $\lim^2 \mathbf{A}_{\aleph_3} \neq 0$ can be constructed to consist of countably supported functions.

Questions

The most pressing questions about $\mathbf{A}_\kappa$ concern the situation below $\aleph_\omega$.

Question

Is it true that, for all $n \geq 2$, we have $\lim^n \mathbf{A}_{\aleph_n} \neq 0$?

This is true for $n = 1$, or if $\mathbf{A}_{\aleph_n}$ is replaced by $\mathbf{A}_{\aleph_n}[\bigoplus_{\omega_n} \mathbb{Z}]$. One promising approach to this question involves higher-dimensional analogues of Todorcevic's walks on ordinals method.

Question

Given $n < m < \omega$, is it consistent that $\lim^n \mathbf{A}_{\aleph_m} = 0$? What about $\lim^n \mathbf{A}_{\aleph_m}[H] = 0$ for all abelian groups H?

A positive answer would be a higher-dimensional analogue of the result that consistently there are no coherent $\aleph_2$-Aronszajn trees. We expect a positive answer, but this seems to require new ideas.

IV. Applications

Additivity of strong homology

Recall that, if there is $n > 0$ for which $\lim^n \mathbf{A} \neq 0$, then strong homology fails to be additive, even on the class of closed subspaces of Euclidean space. It turns out that obtaining a model in which $\lim^n \mathbf{A} = 0$ for all $n > 0$ removed all obstacles to the additivity of strong homology on a broad class of spaces.

Theorem (Bannister-Bergfalk-Moore '23 [2], Bannister, '23 [1])

After adding either a weakly compact number of Hechler reals or $\beth_\omega$-many Cohen reals, strong homology is additive on the class of locally compact separable metric spaces.

A ZFC counterexample to the additivity of strong homology was found by Prasolov. A simpler example was recently found by Bergfalk–LH.

Condensed mathematics

Condensed mathematics is a framework for applying algebraic tools to the study of algebraic structures carrying topologies, developed recently by Dustin Clausen and Peter Scholze. Classical categories of such objects, e.g., TopAb, are poorly behaved algebraically. Condensed mathematics aims to solve this by embedding these classical categories into richer categories with better algebraic properties.

Condensed abelian groups

Let ED denote the category of extremally disconnected compact Hausdorff spaces. A *condensed abelian group* is a contravariant functor $T : \mathrm{ED} \to \mathrm{Ab}$ such that

- $T(\emptyset) = *$;
- for all $S_0, S_1 \in \mathrm{ED}$, $T(S_0 \sqcup S_1) = T(S_0) \times T(S_1)$.

If A is a topological abelian group, then one obtains a condensed abelian group $\underline{A}$ by setting $\underline{A}(S) = \mathrm{Cont}(S, A)$ for all $S \in \mathrm{ED}$. This describes a fully faithful embedding of the category of compactly generated topological abelian groups into the category CondAb of condensed abelian groups.

Sequential limits

Recall that $\mathrm{Ext}^n(\cdot,\cdot)$ is the n^{th} derived functor of $\mathrm{Hom}(\cdot,\cdot)$. Often when doing computations in condensed mathematics, one considers a sequential limit

$$\ldots \twoheadrightarrow M_2 \twoheadrightarrow M_1 \twoheadrightarrow M_0$$

of countable discrete abelian groups and wants to compute $\mathrm{Ext}^n_{\mathsf{CondAb}}(\varprojlim \underline{M}_i, \underline{N})$ for some discrete abelian group N. In such situations, it would be helpful if we could pull the limit outside, i.e., if

$$\mathrm{Ext}^n_{\mathsf{CondAb}}(\varprojlim \underline{M}_i, \underline{N}) = \varinjlim \mathrm{Ext}^n_{\mathsf{CondAb}}(\underline{M}_i, \underline{N}).$$

The assertion that this can always be done turns out to be equivalent to the assertion that $\lim^n \mathbf{A}[H] = 0$ for all $n > 0$ and all abelian groups H.

Assuming that $\lim^n \mathbf{A}[H] = 0$ for all $n > 0$ and all abelian groups H has various nice foundational consequences for condensed mathematics. For example:

- It implies that the category of separable pro-abelian groups embeds fully faithfully into the category of condensed abelian groups.
- It implies that the classical duality between Banach spaces and Smith spaces extends to a derived internal duality in the condensed setting, at least when restricted to separable nonarchimedean Banach spaces. (It remains an interesting question the extent to which something similar can be said about condensed archimedean Banach spaces.)

These are prominent examples on a growing list of "nice" statements that are consistent but require $2^{\aleph_0} > \aleph_\omega$, providing some mild contrast with Osofsky's earlier quote about the "upsetting consequences" of this hypothesis.

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